- A field $(k,+,\times)$ is a set with two operations: $(k,+)$ abelian group with identity 0, $(k^* = k-\{0\},\times)$ abelian group with identity 1; distributive law $a(b+c)=ab+ac$.

Lec. 6 Examples: $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ (characteristic 0: $\underbrace{1+\dots+1}_{n} = n\cdot 1 \neq 0$), $\mathbb{F}_p = \mathbb{Z}/p$ p prime (char. $= p$).

Axler ch. 1

- A vector space over k is a set V with addition $+: V\times V \to V$ ($(V,+)$ abelian group, $0\in V$) scalar mult. $k\times V \to V$ associative, distributive.

Ex: k^n, $k[x]$, … Subspace: $W\subset V$ closed under $+,\times$.

- $\operatorname{span}(v_1,\dots,v_n) = \{\sum a_i v_i \mid a_i \in k\} \subset V$, say (v_i) span V if $\operatorname{span}(v_i) = V$.

Axler ch. 2 Say (v_i) are linearly independent if $a_1v_1+\dots+a_nv_n = 0 \Rightarrow a_i = 0\ \forall i$.

Lec. 7 Basis = linearly independent vectors which span V: $k^n \xrightarrow{\sim} V$, $(a_i) \mapsto \sum a_i v_i$ isomorphism.

- All bases of V have same cardinality $= \dim V$
  Any linearly independent set can be completed to a basis.
- $\operatorname{Hom}(V,W)$ = linear maps $\varphi: V\to W$, $\varphi(u+v)=\varphi(u)+\varphi(v)$, $\varphi(\lambda u) = \lambda\varphi(u)$.
  This is a vector space.

Axler ch. 3

- Given bases $(v_j)_{j=1..n}$ of V, $(w_i)_{i=1..m}$ of W, represent $v = \sum x_j v_j \in V$ by column $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $\varphi \in \operatorname{Hom}(V,W)$ by matrix $A=(a_{ij})$ whose columns represent $\varphi(v_j)$ in basis (w_i), $\varphi(v_j) = \sum_i a_{ij} w_i$.

Lec. 8 Then $\varphi(v)$ is represented in basis (w_i) by column vector $Y = AX$.

Change of basis: $P = (p_{ij}) = M(\mathrm{id}, (v_i'), (v_i))$ ie. $v_j' = \sum p_{ij} v_i$, then for $\varphi: V\to V$, $M(\varphi, (v_i')) = A' = P^{-1}AP$

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \text{basis}\uparrow\simeq & & \uparrow\simeq \text{basis} \\ k^n & \xrightarrow{A} & k^m \end{array}$$

- $V \simeq W_1 \oplus \dots \oplus W_n$ direct sum decomp^n if $\begin{cases} W_i \text{ span } V: \forall v\in V\ \exists w_i \in W_i \text{ st. } v = w_1+\dots+w_n \\ W_i \text{ independent}: w_1+\dots+w_n = 0,\ w_i\in W_i \Rightarrow w_i = 0\ \forall i. \end{cases}$

  ie. $\varphi: \oplus W_i \xrightarrow{\sim} V$ is an isomorphism. $(w_i) \mapsto \sum w_i$
- V finite dim. $\Rightarrow V = W_1\oplus W_2$ iff $W_1\cap W_2 = \{0\}$ and $\dim W_1 + \dim W_2 = \dim V$.
- dim/rank formula: V, W finite dim., $\varphi\in\operatorname{Hom}(V,W) \Rightarrow \dim V = \dim\ker\varphi + \underbrace{\dim \operatorname{Im}\varphi}_{=\operatorname{rank}(\varphi)}$.
- $\exists$ bases (v_i) of V, (w_j) of W st. $M(\varphi) = \left(\begin{array}{c|c} I_{r\times r} & 0 \\ \hline 0 & 0 \end{array}\right)$ $\}\operatorname{Im}\varphi$, $\underbrace{\quad}_{\ker\varphi}$

Lec. 9 - Dual: $V^* = \operatorname{Hom}(V,k)$.

(e_i) basis of V (finite dim) $\Rightarrow$ dual basis (e_i^*) of V^* st. $e_i^*(e_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$.

$V \xrightarrow{\sim} V^{**}$, $v \mapsto \mathrm{ev}_v : \left(\begin{array}{c} V^* \to k \\ \ell \mapsto \ell(v) \end{array}\right)$ is an isomorphism if $\dim V < \infty$ (injective if $\dim V = \infty$).

The annihilator of $U\subset V$ is $\operatorname{Ann}(U) = \{\ell\in V^* \mid \ell(u) = 0\ \forall u\in U\}$; $\dim\operatorname{Ann}(U) = n - \dim U$.

The transpose of $\varphi\in\operatorname{Hom}(V,W)$ is $\varphi^t: W^*\to V^*$, $\varphi^t(\ell) = \ell\circ\varphi$

$\ker\varphi^t = \operatorname{Ann}(\operatorname{Im}\varphi)$, $\operatorname{Im}\varphi^t = \operatorname{Ann}(\ker\varphi)$ if $\dim < \infty$, $M(\varphi^t, (f_j^*), (e_i^*)) = M(\varphi)^T$

- Quotient: $U \subset V$ subspace $\Rightarrow V/U = \{\text{cosets } v+U\}$ is a vector space.
  $V \xrightarrow{q} V/U$, $v \mapsto v+U$ is surjective with kernel $= U$.
  $V \xrightarrow{\varphi} W$ factors through V/U ($q \searrow V/U \dashrightarrow \exists \bar{\varphi}$) iff $U \subset \ker \varphi$.

Axler ch. 5 Lec. 10

- $W \subset V$ is an invariant subspace for $\varphi \in \mathrm{Hom}(V,V)$ if $\varphi(W) \subset W$.
  Ex. $\ker(\varphi)$, $\mathrm{Im}(\varphi)$; eigenspaces $\ker(\varphi - \lambda I)$.
- if $V = \oplus V_i$, V_i invariant for $\varphi \Rightarrow \exists$ basis where $M(\varphi) = \left(\begin{array}{c|c} \varphi|_{V_1} & 0 \\ \hline 0 & \varphi|_{V_2} \end{array}\right)$
  A basis of eigenvectors $v_i \in V$, $v_i \neq 0$, $\varphi(v_i) = \lambda_i v_i \Leftrightarrow \varphi$ diagonalizable, $M_{(\varphi_{(v_i)})} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.
- Eigenvectors of φ for distinct eigenvalues are linearly indep^t

Lec. 11

- If k is algebraically closed (eg. $\mathbb{C}$) then any linear op. $\varphi \in \mathrm{Hom}(V,V)$ has an eigenvector.
  Corollary: $\exists$ basis in which $M(\varphi)$ is upper triangular $\begin{pmatrix} * & & \\ & \ddots & \\ 0 & & * \end{pmatrix}$
  $\lambda \in k$ is an eigenvalue of $\varphi \Leftrightarrow (\varphi - \lambda)$ not invertible $\Leftrightarrow \lambda$ appears on diagonal in a triangular matrix representing φ.

Axler ch. 8

- The generalized kernel $g\ker(\varphi) = \ker(\varphi^N)$ $\forall N$ large (eg. $\geq \dim V$).
  φ is nilpotent if $\varphi^N = 0$; $\ker(\varphi) \subset \ker(\varphi^2) \subset \dots$ $\exists$ basis s.t. $M(\varphi)$ block diagonal with blocks $\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$
- generalized eigenspaces $V_\lambda = g\ker(\varphi - \lambda) = \ker(\varphi - \lambda)^N$ are linearly independent invariant subspaces.
- if k is alg. closed then $V =$ direct sum $\oplus V_\lambda$ of the gen^d eigenspaces of φ.
  This gives the Jordan normal form: $M(\varphi)$ block diagonal with blocks $\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$

Lec. 12

(φ diagonalizable $\Leftrightarrow$ all blocks have size 1).

- characteristic polynomial of φ: $\chi_\varphi(x) = \det(xI - \varphi) = \prod_{\lambda_i} (x - \lambda_i)^{n_i}$, $n_i = \mathrm{mult}(\lambda_i) = \dim V_{\lambda_i}$.
  minimal polynomial: $\mu_\varphi(x) = \prod (x - \lambda_i)^{m_i}$, $m_i = \min\{m \mid V_{\lambda_i} = \ker(\varphi - \lambda_i)^m\}$ = size of largest Jordan block in V_{λ_i}.
- $p(\varphi) = 0$ iff $\mu_\varphi \mid p(x)$. In particular $\mu_\varphi \mid \chi_\varphi$.
  φ diagonalizable $\Leftrightarrow m_i = 1$ $\forall i$.
- Over $\mathbb{R}$, $\varphi : V \to V$ need not have eigenvectors, but by considering $V_{\mathbb{C}} = V \times V = \{v + iw \mid v, w \in V\}$ and $\varphi_{\mathbb{C}} : V_{\mathbb{C}} \to V_{\mathbb{C}}$, $\varphi_{\mathbb{C}}(v+iw) = \varphi(v) + i\varphi(w)$ $\Rightarrow$ any real operator has an invariant subspace of dimension 1 (eigenvector!) or 2.

Handout

- Categories have objects, and morphisms $\mathrm{Mor}(A,B)$ $\forall A, B \in \mathrm{ob}\,\mathcal{C}$, with operation = composition.
  Axioms: $\forall A \in \mathrm{ob}\,\mathcal{C}$, $\exists\, id_A \in \mathrm{Mor}(A,A)$, $f \circ id_A = id_B \circ f = f$; associativity $(f \circ g) \circ h = f \circ (g \circ h)$.
  Ex: sets, groups, vector spaces/k

Lec. 14

- A functor $F : \mathcal{C} \to \mathcal{D}$ assigns
  - to each $X \in \mathrm{ob}\,\mathcal{C}$, $F(X) \in \mathrm{ob}\,\mathcal{D}$
  - to $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$, $F(f) \in \mathrm{Mor}_{\mathcal{D}}(F(X), F(Y))$

  s.t. $F(id_X) = id_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$. (contravariant functors = reverse dir^n of morphisms)
- Natural transformation t between functors $F, G : \mathcal{C} \to \mathcal{D}$
  = for each $X \in \mathrm{ob}\,\mathcal{C}$, $t_X \in \mathrm{Mor}_{\mathcal{D}}(F(X), G(X))$ s.t. $\forall$ $\begin{array}{c} X \\ \downarrow f \\ Y \end{array}$, $\begin{array}{ccc} F(X) & \xrightarrow{t_X} & G(X) \\ \downarrow F(f) & & \downarrow G(f) \\ F(Y) & \xrightarrow{t_Y} & G(Y) \end{array}$ commutes.

Axler ch.9A Lec. 14

- A bilinear form on V is $b: V \times V \to k$, linear in each input: $b(u+v,w) = b(u,w)+b(v,w)$, $b(\lambda u, v) = \lambda b(u,v)$ etc.
  b is symmetric if $b(u,v) = b(v,u)$, skew-symmetric if $b(u,v) = -b(v,u)$.
- $B(V) = \{\text{bilinear } b: V\times V \to k\} \xrightarrow{\sim} \mathrm{Hom}(V, V^*)$ (isom. of vector spaces)
  $b \longmapsto \varphi_b : v \longmapsto (b(v,\cdot): V \to k)$
  $\mathrm{rank}(b) = \mathrm{rank}(\varphi_b)$, b is nondegenerate if $\varphi_b : V \xrightarrow{\sim} V^*$ isomorphism.
- in a basis (e_i) of V, b is represented by a matrix $B = (b_{ij}) = (b(e_i, e_j))$.
  if $u = \sum x_i e_i$, $v = \sum y_i e_i$ are represented by column vectors X, Y, $b(u,v) = X^T B Y$.

Lec. 15

- the orthogonal of $S \subset V$ for b is $S^\perp = \{v \in V / b(v,w) = 0 \ \forall w \in S\} = \mathrm{Ker}(V \to S^*)$, $v \mapsto \varphi_b(v)|_S$
  If b is nondegenerate then $\dim S^\perp = \dim V - \dim S$

Axler ch.6

  If b is an inner product then $S \cap S^\perp = \{0\}$ and $V = S \oplus S^\perp$.

- A real inner product $\langle \cdot,\cdot \rangle : V \times V \to \mathbb{R}$ is a symmetric definite positive bilinear form.
  Cauchy-Schwarz ineq: $\langle u, v\rangle \leq \|u\|\,\|v\|$. ↳ $\langle u,u\rangle = \|u\|^2 > 0 \ \forall u \neq 0$.

Over $\mathbb{C}$, we consider Hermitian inner products $\langle \cdot,\cdot \rangle : V \times V \to \mathbb{C}$, not quite bilinear: $\langle \lambda u, v\rangle = \bar\lambda \langle u, v\rangle$
require Hermitian-symmetric $\langle v, u\rangle = \overline{\langle u, v\rangle}$, and definite positive $\langle u,u\rangle = \|u\|^2 > 0 \ \forall u \neq 0$.
The map $V \to V^*$ induced by such $\langle\cdot,\cdot\rangle$ is $\mathbb{C}$-antilinear: $\varphi(\lambda u) = \bar\lambda \varphi(u)$.

- Every finite dimensional inner product space (over $\mathbb{R}$ or $\mathbb{C}$) has an orthonormal basis $(e_1, \dots, e_n)$ st. $\langle e_i, e_j\rangle = \delta_{ij}$. (build by induction eg. using Gram-Schmidt).

Axler ch.7

- Let $V, \langle\cdot,\cdot\rangle$ inner product space (over $\mathbb{R}$ or $\mathbb{C}$), $T: V \to V$ linear operator.
  The adjoint operator $T^*: V \to V$ satisfies $\langle v, Tw\rangle = \langle T^* v, w\rangle \ \forall v, w \in V$.
  (corresponds to the transpose of T via $V \xrightarrow{\sim}_{\varphi} V^*$; over $\mathbb{C}$: complex conjugate of T^t).
  In an orthonormal basis, $M(T^*) = M(T)^t$ (real case) or $\overline{M(T)}^t$ (complex Hermitian case)
  $\mathrm{Ker}(T^*) = \mathrm{Im}(T)^\perp$ and vice-versa.

Lec. 16

- $T: V \to V$ is self-adjoint if $T^* = T$

Lec. 17

  T is orthogonal (unitary over $\mathbb{C}$) if $T^* = T^{-1}$, ie. $\langle Tu, Tv\rangle = \langle u, v\rangle \ \forall u, v \in V$.
  ($\Leftrightarrow$ T maps orthonormal basis to orthonormal basis)

- If $S \subset V$ is invariant under a self-adjoint/orthogonal/unitary operator then so is $S^\perp$.

$\Rightarrow$ Spectral theorem (real and complex versions):

- If $T: V \to V$ is self adjoint then T is diagonalizable, with real eigenvalues, and can be diagonalized in an orthonormal basis.
- If $T: V \to V$ is orthogonal for a real inner product, then V is a direct sum of orthogonal invariant subspaces of dim. 1 or 2, with T acting by ± 1 on the 1-dim. pieces, rotations on 2-dim. pieces.
- If $T: V \to V$ is unitary for a Hermitian inner product, then T is diagonalizable in an orthonormal basis, with eigenvalues $|\lambda_i| = 1$.
  More generally if $T: V \to V$ (Hermitian) is normal ie. $TT^* = T^*T \Rightarrow T$ is diagonalizable in an orthonormal basis.

- Besides inner products, one can also consider arbitrary nondegenerate symmetric bilinear forms (without assuming positivity); eg. over $\mathbb{R}$ (resp. $\mathbb{C}$), $\exists$ orthogonal basis st. $b(e_i,e_j) = \begin{cases} \pm 1 & i=j \\ 0 & i\neq j \end{cases}$ (resp. $b(e_i,e_j)=\delta_{ij}$); or skew-symmetric bilinear forms.

Handout Lec. 18

- Tensor product: $V\otimes W$ vector space, with a bilinear map $V\times W \to V\otimes W$, $(v,w)\mapsto v\otimes w$, st. bilinear maps $V\times W \xrightarrow{b} U$ correspond to linear maps $V\otimes W \xrightarrow{\varphi} U$ $(\varphi(v\otimes w)=b(v,w))$

  Elements of $V\otimes W$ are finite linear combinations $\sum v_i\otimes w_i$

  If (e_i) basis of V and (f_j) basis of W, then $(e_i\otimes f_j)$ basis of $V\otimes W$.

- $V^*\otimes W \cong \mathrm{Hom}(V,W)$, by mapping $\ell\otimes w \in V^*\otimes W$ to $(v\mapsto \ell(v)w)\in \mathrm{Hom}(V,W)$.
- the trace $\mathrm{tr}(T:V\to V)=\sum\lambda_i\in k$ can be defined by $\mathrm{Hom}(V,V)\xrightarrow{\sim} V^*\otimes V \to k$, $\ell\otimes v\mapsto \ell(v)$

Lec. 19

- multilinear maps $V_1\times\dots\times V_n\to U \Leftrightarrow$ linear maps $V_1\otimes\dots\otimes V_n\to U$.
- $V^{\otimes n}=V\otimes\dots\otimes V$ contains subspaces

  $\mathrm{Sym}^n(V)$ = symmetric tensors ($\leftrightarrow$ symmetric multilinear maps) $v_{\sigma(1)}\dots v_{\sigma(n)}=v_1\dots v_n$

  $\Lambda^n(V)$ = exterior powers: alternating tensors $v_{\sigma(1)}\wedge\dots\wedge v_{\sigma(n)}=(-1)^\sigma v_1\wedge\dots\wedge v_n$.

Lec. 20

- if $\dim V=n$ then $\Lambda^n V$ has dim. 1; for $T:V\to V$, $\Lambda^n T:\Lambda^n V\to\Lambda^n V$ is multiplication by a scalar, the determinant $\det(T)\in k$.

- The theory of modules over a ring $(R,+,\times)$ (elements need not have multiplicative inverses) is more complicated than that of vector spaces.

  Finitely generated modules need not have a basis; those that do are called free.

- $\mathbb{Z}$-modules $\Leftrightarrow$ abelian groups.

Lec. 21 (parts of Artin ch. 14)

Every finitely generated $\mathbb{Z}$-module M with generators $(e_1..e_n)$ is a quotient of $\mathbb{Z}^n$ $\left(\varphi:\begin{array}{l}\mathbb{Z}^n\twoheadrightarrow M\\ (a_i)\mapsto\sum a_ie_i\end{array}\right)$ and $\ker(\varphi)\subset\mathbb{Z}^n$ is itself a free module, i.e. $\exists\, T:\mathbb{Z}^m\to\mathbb{Z}^n$ st. $M\cong\mathbb{Z}^n/\mathrm{Im}\,T$

→ via linear algebra over $\mathbb{Z}$, one finds:

Every finitely generated abelian group is $\cong \mathbb{Z}^r\times\mathbb{Z}/n_1\times\dots\times\mathbb{Z}/n_k$ for some $r, n_1..n_k$.