Lec. 1 • A group $(G,\cdot)$ is a set with an operation $\cdot: G\times G\to G$ st. (1) $\exists e\in G$ identity s.t. $eg=ge=g\ \forall g\in G$,
Artin ch. 2 (2) $\forall g\in G\ \exists$ inverse $g^{-1}\in G$ st. $gg^{-1}=g^{-1}g=e$, (3) associativity $(ab)c=a(bc)\ \forall a,b,c\in G$.

• A group is abelian if $\cdot$ is commutative ($ab=ba\ \forall a,b\in G$)

• Ex: $(\mathbb{Z},+)$, $(\mathbb{Z}/n,+)$, $(\mathbb{C}^*,\times)$, symmetric group S_n; $GL_n(\mathbb{R})$ etc.; products $G\times H$, $\mathbb{Z}^n$, ...

Lec. 2 • Like sets, groups can be finite ($\mathbb{Z}/n$, S_n, ...), countable ($\mathbb{Z}$, $\mathbb{Z}^n$, $\mathbb{Q}$, ...), uncountable ($\mathbb{R}$ ...)

• $H\subset G$ is a subgroup if $e\in H$, $a\in H\Rightarrow a^{-1}\in H$, $a,b\in H\Rightarrow ab\in H$. $|H|$ divides $|G|$.
H,H' subgroups of $G\Rightarrow H\cap H'$ is a subgroup of G.
All subgroups of $(\mathbb{Z},+)$ are $\mathbb{Z}n=\{mn/m\in\mathbb{Z}\}$ for some $n\geq 0$.

• A homomorphism $\varphi: G\to H$ is a map st. $\varphi(ab)=\varphi(a)\varphi(b)\ \forall a,b\in G$. ($\Rightarrow \varphi(a^{-1})=\varphi(a)^{-1}$)
isomorphism = bijective homomorphism, automorphism = isom $G\xrightarrow{\sim}G$. $(Aut(G),\circ)$ is a group.

• The kernel of $\varphi: G\to H$: $\ker(\varphi)=\{g\in G/\varphi(g)=e_H\}$ subgroup of G. φ injective $\Leftrightarrow \ker\varphi=\{e\}$
The image of φ: $Im(\varphi)=\{\varphi(g)/g\in G\}\subset H$ subgroup of H. φ surjective $\Leftrightarrow Im\,\varphi=H$.

Lec. 3 • Given $a\in G$, $\varphi: \mathbb{Z}\to G$, $k\mapsto a^k$ is a homomorphism with $im(\varphi)=\langle a\rangle$ subgp generated by a.
$\ker(\varphi)=\mathbb{Z}n$ where $n=$ order of $a=\min\{n>0 \text{ st. } a^n=e\}$.
Hence the cyclic group $\langle a\rangle$ is $\simeq\mathbb{Z}/n$ if a has order n, $\simeq\mathbb{Z}$ if infinite order.
($a_1,\dots,a_k$ generate G if every element of G is a product of a_i and their inverses).

• A subgroup $H\subset G$ determines an equivalence relation (axioms: $a\sim a$; $a\sim b\Leftrightarrow b\sim a$; $\left.\begin{matrix}a\sim b\\ b\sim c\end{matrix}\right\}\Rightarrow a\sim c$)
Lec. 4 $a\sim b$ iff $a^{-1}b\in H$, whose equivalence classes are the (left) cosets $aH=\{ah/h\in H\}$.
The quotient set: $G/H=\{\text{cosets } aH\}$. The index of H: $(G:H)=|G/H|=\frac{|G|}{|H|}$ if finite.

• If G is finite: $H\subset G$ subgroup $\Rightarrow |H|$ divide $|G|$; $a\in G\Rightarrow \text{ord}(a)\,|\,|G|$; $|G|=p$ prime $\Rightarrow G\simeq\mathbb{Z}/p$.

• A subgroup $H\subset G$ is normal $\Leftrightarrow aH=Ha\ \forall a\in G\Leftrightarrow aHa^{-1}=H\ \forall a\in G$.
(left cosets = right cosets) (conjugate subgroups)

• The operation $(aH)\cdot(bH)=abH$ makes G/H a group iff H is a normal subgroup.

• $\forall\varphi: G\to H$ homomorphism, $\ker(\varphi)=K$ is a normal subgroup of G, and $Im(\varphi)\simeq G/K$.

Lec. 5 If φ is surjective, we have an exact sequence $\{1\}\to K\xrightarrow[\text{injective}]{i}G\xrightarrow[\text{surjective}]{\varphi}H\to\{1\}$ $Im(i)=\ker(\varphi)$.

Ex: $\{1\}\to H\xrightarrow[\text{normal}]{}G\to G/H\to\{1\}$; $0\to\mathbb{Z}/m\xrightarrow{i}\mathbb{Z}/mn\to\mathbb{Z}/n\to 0$ ($\mathbb{Z}/mn\simeq\mathbb{Z}/m\times\mathbb{Z}/n$ iff $\gcd(m,n)=1$)

$\{e\}\to\mathbb{Z}/3\to S_3\xrightarrow{\text{sign}}\mathbb{Z}/2\to\{e\}$

A homomorphism $G\xrightarrow{\varphi}H$ factors through $G\to G/K\xrightarrow{\bar\varphi}H$ iff $K\subset\ker\varphi$

• G is simple if its only normal subgroups are $\{e\}$ and itself. Ex: $\mathbb{Z}/p$ p prime; A_n $n\geq 5$.

• Ex: the center $Z(G)=\{z\in G/zg=gz\ \forall g\in G\}$ is a normal subgroup (abelian: $zz'=z'z$)

• Ex: the commutator subgroup $[G,G]=\{\prod_{\text{finite}}[a_i,b_i]\}$, where $[a,b]=aba^{-1}b^{-1}$, is normal, and $G/[G,G]=Ab(G)$ (abelianization) largest abelian quotient of G. $\forall G\xrightarrow{\varphi}H$, H abelian factors $G\to Ab(G)\xrightarrow{\bar\varphi}H$.

Lec. 21 • Every finitely generated abelian group is $\simeq\mathbb{Z}^r\times\mathbb{Z}/n_1\times\dots\times\mathbb{Z}/n_k$ for some $r,n_1,\dots,n_k$.
Artin 14.7

Artin 6.7–6.12 • **Group actions**: G-action on set S: $G \times S \to S$, $(g,s) \mapsto g\cdot s$ s.t. $e\cdot s = s$ $\forall s \in S$, $(gh)\cdot s = g\cdot(h\cdot s)$ ($\Leftrightarrow$ homom. $\rho: G \to \mathrm{Perm}(S)$) ②

faithful if ρ injective; transitive if $\forall s,t \in S$ $\exists g$ s.t. $g\cdot s = t$ (ie: 1 orbit)

Lec. 22 • The orbit of $s \in S$ is $\mathcal{O}_s = G\cdot s = \{g\cdot s \mid g \in G\}$. These form a partition $S = \coprod$ orbits.

The stabilizer of s is $\mathrm{Stab}(s) = \{g \in G \mid g\cdot s = s\}$ subgroup of G.

Elements in same orbit have conjugate stabilizer subgroups $\mathrm{Stab}(g\cdot s) = g\,\mathrm{Stab}(s)\,g^{-1} \subset G$.

Orbit-stabilizer: if $H = \mathrm{Stab}(s)$, then $G/H \simeq \mathcal{O}_s$ bijection, in particular $|\mathcal{O}_s| \cdot |\mathrm{Stab}\, s| = |G|$.
$gH \mapsto g\cdot s$

• Burnside's lemma (G, S finite): let $S^g = \{s \in S \mid gs = s\}$ fixed points of $g \in G$, then $\#\text{orbits} = \frac{1}{|G|}\sum_{g\in G} |S^g|$

Artin ch. 7 • G acts on itself by left multiplication. This gives $G \hookrightarrow \mathrm{Perm}(G)$, hence:
every finite group G is isomorphic to a subgroup of S_n, $n = |G|$.

• G acts on itself by conjugation: g acts by $h \mapsto ghg^{-1}$.

Lec. 24 orbits = conjugacy classes; $\mathrm{Stab}(h) = \{g \in G \mid gh = hg\} = Z(h)$ centralizer of h.

Hence: $|G| = \sum_{C \subset G \text{ conj. classes}} |C|$, where for each conj. class $|C_h| = \frac{|G|}{|Z(h)|}$ divides $|G|$. (class eq^n of G)

• For p-groups ($|G| = p^k$), the class equation $\Rightarrow |Z(G)| \geq p$ (number of conj. classes of size 1)

Hence: • $|G| = p^2$, p prime $\Rightarrow G$ is abelian ($\simeq \mathbb{Z}/p \times \mathbb{Z}/p$ or $\mathbb{Z}/p^2$)

• 5 isom. classes of groups of order 8: $\mathbb{Z}/8$, $\mathbb{Z}/4 \times \mathbb{Z}/2$, $(\mathbb{Z}/2)^3$, D_4, quaternion group.

Lec. 23: • $G \subset SO(3)$ finite subgp $\Rightarrow$ by considering the action of G on its poles (unit vectors along rotation axes), $G \simeq$ one of $\mathbb{Z}/n$, D_n (regular n-gon), A_4 (tetrahedron), S_4 (cube), A_5 (dodecahedron/icosahedron)

Lec. 24: • The symmetric group S_n is generated by transpositions (ij), in fact by $s_i = (i\ i+1)$.

• $\forall \sigma \in S_n$ $\exists$ unique decomp of σ as product of disjoint cycles $(a_1 \dots a_k)$.
$\sigma, \tau \in S_n$ are in same conjugacy class iff they have the same cycle lengths.

• the alternating group $A_n = \ker(\mathrm{sign}: S_n \to \mathbb{Z}/2) = \{\text{products of even \# of transpositions}\}$

Lec. 25: A conjugacy class in S_n which consists of even permutations is either 1 or 2 conj. classes in A_n; it splits into 2 iff the centralizer $Z(\sigma) \subset A_n$ ($\Leftrightarrow$ cycle lengths of σ are all odd & distinct).

• A_n is simple for $n \geq 5$ (A_4 isn't: $\{id, (ij)(kl)\} \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ is normal in A_4 and S_4).

Lec. 26: • Sylow theorems: $|G| = p^e m$, $p \nmid m$ $\Rightarrow$ a Sylow p-subgroup of G is a subgp. of order p^e.

Thm 1: $\forall p$ prime $| |G|$, G contains a Sylow p-subgroup. ($\rightsquigarrow$ consequence: G contains an elt of order p)

Thm 2: all Sylow p-subgroups of G are conjugates of each other, and every subgroup of order p^k ($k \leq e$) is contained in a Sylow subgroup.

Thm 3: the number s_p of Sylow p-subgroups satisfies $s_p \equiv 1 \bmod p$ and $s_p \mid m = \frac{|G|}{p^e}$.

• If G contains subgroups N, H s.t. $N \cap H = \{e\}$ (eg because $\gcd(|N|, |H|) = 1$) and $|G| = |N| \cdot |H|$, then $\forall g \in G$ $\exists$ unique $n \in N$, $h \in H$ s.t. $g = nh$.
If N and H are both normal in G then $G \simeq N \times H$. If N is normal but not H, we have a semidirect product $N \rtimes_\varphi H$, $\varphi: H \to \mathrm{Aut}(N)$ given by conjugation inside G.
$$(n,h)\cdot(n',h') = (n\,\varphi(h)(n'),\ hh')$$

Lec. 28
- given $H \subset G$ (eg. p-Sylow), the number of conjugate subgroups $gHg^{-1} \subset G$ (eg. all p-Sylows) equals $|G/N(H)|$, $N(H)$ normalizer $= \{g \in G \mid gHg^{-1} = H\}$ (largest subgp of G st. H is normal inside N).
- Example: $|G| = 15 \Rightarrow$ Sylow subgroups of order 3 and 5 are normal ($s_3 = s_5 = 1$) $\Rightarrow G \simeq \mathbb{Z}/3 \times \mathbb{Z}/5$.
  $|G| = 21 \Rightarrow s_3 \in \{1, 7\}$, $s_7 = 1$, so either $G \simeq \mathbb{Z}/3 \times \mathbb{Z}/7$ or semidirect product $\mathbb{Z}/7 \rtimes \mathbb{Z}/3$.
  $|G| = 12 \Rightarrow$ 1 or 3 2-Sylows, 1 or 4 3-Sylows, one of these is normal $\Rightarrow$ 5 isom. classes:
  $\mathbb{Z}/4 \times \mathbb{Z}/3$, $(\mathbb{Z}/2)^2 \times \mathbb{Z}/3$, A_4, D_6, $\mathbb{Z}/3 \rtimes \mathbb{Z}/4$.

Lec. 29
- The free group $F_n = \langle a_1 \dots a_n \rangle = \{$all reduced words $a_{i_1}^{m_1} \dots a_{i_k}^{m_k}\}$
  (words in $a_i^{\pm 1}$ never simplify except $a_i a_i^{-1} = a_i^{-1} a_i = 1$)
- Any group G with n generators $g_1 \dots g_n$ is a quotient of F_n, via $\varphi: F_n \twoheadrightarrow G$, $a_i \mapsto g_i$
  G is finitely presented if $\ker(\varphi)$ is generated by a finite set $r_1, \dots, r_k$ & their conjugates
  Write $G \simeq \langle g_1 \dots g_n \mid r_1 \dots r_k \rangle = F_n / \langle$normal subgp gen^d by conjugates of $r_j \rangle$.
- The Cayley graph of G w/ generators g_i: vertices = elements of G
  edges: connect g to $g \cdot g_i$ $\forall g \in G$, $\forall g_i$
- A normal form for elements of $G = \langle g_1 \dots g_n \mid r_1 \dots r_k \rangle$ is a set of words in $g_1^{\pm 1} \dots g_n^{\pm 1}$ st. every element of G appears exactly once among these.

Lec. 30
- Ex: $S_n \simeq \langle s_1, \dots, s_{n-1} \mid s_i^2 = 1,\ s_i s_j = s_j s_i \ \ |i-j| \geq 2,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \rangle$.
  $SL_2(\mathbb{Z})$ is gen^d by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z}) / \{\pm I\} = \langle S, T \mid S^2, (ST)^3 \rangle$

Lec. 30
Artin ch. 10
Fulton-Harris ch. 1-2
- A representation of G is a vector space V on which G acts by linear operators; ie. $\rho: G \to GL(V)$ homomorphism
- A subrepresentation is a subspace $W \subset V$ invariant under G: $g(W) = W$ $\forall g \in G$.
  V is irreducible if has no nontrivial subrepresentations
- G finite, V finite dim$_{\mathbb{C}}$: each $g: V \to V$ has finite order, $g^n = Id \Rightarrow$ diagonalizable, $\lambda_j = e^{2\pi i k/n}$

Lec. 31
- if G is abelian, all operators $g: V \to V$ are simultaneously diagonalizable $\Rightarrow$ irred. rep^s are 1-dim^l.
  These correspond to elements of the dual group $\hat{G} = \mathrm{Hom}(G, \mathbb{C}^*)$. (Note $\hat{\mathbb{Z}/m}$ is $\simeq \mathbb{Z}/m$)
- a homomorphism of representations is a G-equivariant linear map, ie. $\varphi(gv) = g\varphi(v)$.
- V, W rep^s of $G \Rightarrow$ so are $V \oplus W$, $V \otimes W$ ($g: v \otimes w \mapsto gv \otimes gw$), V^* ($\ell \mapsto \ell \circ g^{-1}$),
  $V^* \otimes W \simeq \mathrm{Hom}(V, W)$ ($\varphi \mapsto g \circ \varphi \circ g^{-1}$). ($\mathrm{Hom}_G(V, W) =$ invariant part $\mathrm{Hom}(V, W)^G$)
- Any $\mathbb{C}$-representation of a finite group G admits an invariant Hermitian inner product, with respect to which G acts by unitary operators.
- V rep. of a finite group (over $\mathbb{C}$), $W \subset V$ invariant subspace $\Rightarrow \exists U \subset V$ invariant st. $V = U \oplus W$.
  Hence: any $\mathbb{C}$-representation of a finite group decomposes into a direct sum of irreducibles.

Lec. 32
- Schur's lemma: V, W irred. rep's of $G \Rightarrow$ any homom. $\varphi \in \mathrm{Hom}_G(V, W)$ is either zero or an isomorphism; and all iso's of an irred. rep. are multiples of id: $\mathrm{Hom}_G(V, V) = \mathbb{C} \cdot id_V$
- Ex: reps. of S_n: trivial rep $U = \mathbb{C}$, σ acts by id; alternating rep: $U' = \mathbb{C}$, σ acts by $(-1)^\sigma$.
  standard rep. (dim. $n-1$): $V = \{(z_1, \dots, z_n) \mid \sum z_i = 0\} \subset \mathbb{C}^n$, σ acts by permuting coords: $e_i \mapsto e_{\sigma(i)}$.
  U, U', V are the only irred. reps of S_3.

Lec·33 :
- The key tool to study representations is the character $\chi_V : G \to \mathbb{C}$, $\chi_V(g) = \mathrm{tr}(g : V \to V)$
  (In terms of eigenvalues, $\mathrm{tr}(g) = \sum \lambda_i$, and $\mathrm{tr}(g^k) = \sum \lambda_i^k$, so χ_V recovers all symmetric polynomial expressions in the λ_i, hence the λ_i as unordered tuple).
  $\chi_V : G \to \mathbb{C}$ is a class function, ie. $\chi_V(hgh^{-1}) = \chi_V(g)$.
- $\chi_{V \oplus W} = \chi_V + \chi_W$, $\chi_{V \otimes W} = \chi_V \chi_W$, $\chi_{V^*} = \overline{\chi_V}$, $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V}\,\chi_W$.
- for a permutation rep. (G acting on $S \rightsquigarrow G$ acts on V with basis $(e_s)_{s \in S}$, $g \cdot e_s = e_{g \cdot s}$)
  $\chi(g) = \#\{s \in S / g \cdot s = s\} = |S^g|$.

Lec·34
- Character table of G = list, for each irred. rep. V_i, the value of χ_{V_i} on each conjugacy class.
- $\varphi = \frac{1}{|G|} \sum_{g \in G} g : V \to V$ projection onto $V^G = \{v \in V / gv = v \ \forall g\}$, so $\dim(V^G) = \mathrm{tr}(\varphi) = \frac{1}{|G|} \sum_g \chi_V(g)$
- $H(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g)$ Hermitian inner product on $\mathbb{C}_{\mathrm{class}}(G) = \{\text{class functions } G \to \mathbb{C}\}$
  then $\dim \mathrm{Hom}_G(V, W) = H(\chi_V, \chi_W)$.
- The characters of the irreducible reps of G are an orthonormal basis of $(\mathbb{C}_{\mathrm{class}}(G), H)$.
  In particular the number of irred. reps = number of conjugacy classes
- The multiplicities a_i in the decomposition of a G-rep. W into irreducibles $W \simeq \bigoplus_i V_i^{\oplus a_i}$ are given by $a_i = \dim \mathrm{Hom}_G(V_i, W) = H(\chi_{V_i}, \chi_W)$. Moreover, $H(\chi_W, \chi_W) = \sum a_i^2$.
- The regular repⁿ of G (= permutation rep. for G acting on itself by left multiplication) contains each irred. rep. V_i with multiplicity $= \dim V_i$; therefore $|G| = \sum_i (\dim V_i)^2$.

Lec·35
- These results allow us to find character tables of various groups (eg. S_4, A_4, S_5, A_5) by starting from known representations, considering tensor products, and using $H(\cdot,\cdot)$ pairings and orthogonality to find irreducible pieces & the missing irreducible reps.