Part 1: GROUPS

Lec. 1 / Artin ch. 2

- A group $(G,\cdot)$ is a set with an operation $\cdot: G\times G\to G$ st. (1) $\exists e\in G$ identity s.t. $eg=ge=g\ \forall g\in G$, (2) $\forall g\in G\ \exists$ inverse $g^{-1}\in G$ st. $gg^{-1}=g^{-1}g=e$, (3) associativity $(ab)c=a(bc)\ \forall a,b,c\in G$.
- A group is abelian if $\cdot$ is commutative ($ab=ba\ \forall a,b\in G$)
- Ex: $(\mathbb{Z},+)$, $(\mathbb{Z}/n,+)$, $(\mathbb{C}^*,\times)$, symmetric group S_n, dihedral group D_n
  products $G\times H$, $(g,h)\cdot(g',h')=(gg',hh')$, eg. $\mathbb{Z}\times\dots\times\mathbb{Z}=\mathbb{Z}^n$, etc.

Lec. 2

- Like sets, groups can be finite ($\mathbb{Z}/n$, S_n, ...), countable ($\mathbb{Z}$, $\mathbb{Z}^n$, $\mathbb{Q}$, ...), uncountable ($\mathbb{R}$ ...)
- $H\subset G$ is a subgroup if $e\in H$, $a\in H\Rightarrow a^{-1}\in H$, $a,b\in H\Rightarrow ab\in H$. $|H|$ divides $|G|$.
  H,H' subgroups of $G\Rightarrow H\cap H'$ is a subgroup of G.
  All subgroups of $(\mathbb{Z},+)$ are $\mathbb{Z}n=\{mn\,/\,m\in\mathbb{Z}\}$ for some $n\geq 0$.
- A homomorphism $\varphi: G\to H$ is a map st. $\varphi(ab)=\varphi(a)\varphi(b)\ \forall a,b\in G$. ($\Rightarrow\varphi(a^{-1})=\varphi(a)^{-1}$)
  isomorphism = bijective homomorphism, automorphism = isom $G\xrightarrow{\sim}G$. $(\mathrm{Aut}(G),\circ)$ is a group.
- The kernel of $\varphi: G\to H$: $\ker(\varphi)=\{g\in G\,/\,\varphi(g)=e_H\}$ subgroup of G. φ injective $\Leftrightarrow\ker\varphi=\{e\}$
  The image of φ: $\mathrm{Im}(\varphi)=\{\varphi(g)\,/\,g\in G\}\subset H$ subgroup of H. φ surjective $\Leftrightarrow\mathrm{Im}\,\varphi=H$.

Lec. 3

- Given $a\in G$, $\varphi: \underset{k\mapsto a^k}{\mathbb{Z}\to G}$ is a homomorphism with $\mathrm{im}(\varphi)=\langle a\rangle$ subgp generated by a.
  $\ker(\varphi)=\mathbb{Z}n$ where $n=$ order of $a=\min\{n>0$ st. $a^n=e\}$.
  Hence the cyclic group $\langle a\rangle$ is $\simeq\mathbb{Z}/n$ if a has order n, $\simeq\mathbb{Z}$ if infinite order.
  ($a_1,\dots,a_k$ generate G if every element of G is a product of a_i and their inverses).
- A subgroup $H\subset G$ determines an equivalence relation (axioms: $a\sim a$; $a\sim b\Leftrightarrow b\sim a$; $\left.\begin{matrix}a\sim b\\ b\sim c\end{matrix}\right\}\Rightarrow a\sim c$)

Lec. 4

  $a\sim b$ iff $a^{-1}b\in H$, whose equivalence classes are the (left) cosets $aH=\{ah\,/\,h\in H\}$.
  The quotient set: $G/H=\{\text{cosets } aH\}$. The index of H: $(G:H)=|G/H|=\frac{|G|}{|H|}$ if finite.
- If G is finite: $H\subset G$ subgroup $\Rightarrow|H|$ divide $|G|$; $a\in G\Rightarrow\mathrm{ord}(a)\,|\,|G|$; $|G|=p$ prime $\Rightarrow G\simeq\mathbb{Z}/p$.
- A subgroup $H\subset G$ is normal $\Leftrightarrow aH=Ha\ \forall a\in G\Leftrightarrow aHa^{-1}=H\ \forall a\in G$.
  (left cosets = right cosets) (conjugate subgroups)
- The operation $(aH)\cdot(bH)=abH$ makes G/H a group iff H is a normal subgroup.
- $\forall\varphi: G\to H$ homomorphism, $\ker(\varphi)=K$ is a normal subgroup of G, and $\mathrm{Im}(\varphi)\simeq G/K$.

Lec. 5

  If φ is surjective, we have an exact sequence $\{1\}\to K\xrightarrow[\text{injective}]{i}G\xrightarrow[\text{surjective}]{\varphi}H\to\{1\}$ $\mathrm{Im}(i)=\ker(\varphi)$.
  Ex: $\{1\}\to\underset{\text{normal}}{H\subset G}\to G/H\to\{1\}$; $0\to\mathbb{Z}/m\to\mathbb{Z}/mn\to\mathbb{Z}/n\to 0$ ($\mathbb{Z}/mn\simeq\mathbb{Z}/m\times\mathbb{Z}/n$ iff $\gcd(m,n)=1$)
  $\{e\}\to\mathbb{Z}/3\to S_3\xrightarrow{\text{sign}}\mathbb{Z}/2\to\{e\}$
  A homomorphism $G\xrightarrow{\varphi}H$ factors through $G\to G/K\xrightarrow{\bar\varphi}H$ iff $K\subset\ker\varphi$
- G is simple if its only normal subgroups are $\{e\}$ and itself. Ex: $\mathbb{Z}/p$ p prime; A_n $n\geq 5$.
- Ex: the center $Z(G)=\{z\in G\,/\,zg=gz\ \forall g\in G\}$ is a normal subgroup (abelian: $zz'=z'z$)

Part 2 : LINEAR ALGEBRA

- A field $(k,+,\times)$ is a set with two operations: $(k,+)$ abelian group with identity 0, $(k^* = k-\{0\},\times)$ abelian group with identity 1; distributive law $a(b+c)=ab+ac$.

Lec. 6 Examples: $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ (characteristic 0: $\underbrace{1+\dots+1}_{n} = n\cdot 1 \neq 0$), $\mathbb{F}_p = \mathbb{Z}/p$ p prime (char. $= p$).

Axler ch. 1

- A vector space over k is a set V with addition $+: V\times V \to V$ $(V,+)$ abelian group, $0\in V$
  scalar mult. $k\times V \to V$ associative, distributive.

  Ex: k^n, $k[x]$, ... Subspace: $W\subset V$ closed under $+,\times$.

- $\mathrm{span}(v_1,\dots v_n) = \{\sum a_i v_i \mid a_i \in k\} \subset V$, say (v_i) span V if $\mathrm{span}(v_i) = V$.

Axler ch. 2 Say (v_i) are linearly independent if $a_1v_1+\dots+a_nv_n = 0 \Rightarrow a_i = 0\ \forall i$.

Lec. 7 Basis = linearly independent vectors which span V: $k^n \longrightarrow V$, $(a_i) \mapsto \sum a_i v_i$ isomorphism.

- All bases of V have same cardinality $= \underline{\dim V}$
  Any linearly independent set can be completed to a basis.

- $\mathrm{Hom}(V,W)$ = linear maps $\varphi: V\to W$, $\varphi(u+v)=\varphi(u)+\varphi(v)$, $\varphi(\lambda u) = \lambda\varphi(u)$.
  This is a vector space.

- given bases $(v_j)_{j=1..n}$ of V, $(w_i)_{i=1..m}$ of W, represent $v=\sum x_j v_j \in V$ by column $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Axler ch. 3 and $\varphi \in \mathrm{Hom}(V,W)$ by matrix $A=(a_{ij})$ whose columns represent $\varphi(v_j)$ in basis (w_i), $\varphi(v_j) = \sum_i a_{ij} w_i$.

Lec. 8 Then $\varphi(v)$ is represented in basis (w_i) by column vector $Y = AX$.

Change of basis: $P = (p_{ij}) = M(\mathrm{id}, (v_i'), (v_i))$ ie. $v_j' = \sum p_{ij} v_i$,
then for $\varphi: V\to V$, $M(\varphi, (v_i')) = A' = P^{-1}AP$

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \text{basis}\uparrow\simeq & & \uparrow\simeq \text{basis} \\ k^n & \xrightarrow{A} & k^m \end{array}$$

- $V \simeq W_1 \oplus \dots \oplus W_n$ direct sum decomp^n if $\begin{cases} W_i \text{ span } V: \forall v\in V\ \exists w_i\in W_i \text{ st. } v = w_1+\dots+w_n \\ W_i \text{ independent}: w_1+\dots+w_n = 0,\ w_i\in W_i \Rightarrow w_i = 0\ \forall i. \end{cases}$

  ie. $\varphi: \oplus W_i \longrightarrow V$ is an isomorphism.
  $(w_i) \longmapsto \sum w_i$

- V finite dim. $\Rightarrow V = W_1\oplus W_2$ iff $W_1\cap W_2 = \{0\}$ and $\dim W_1 + \dim W_2 = \dim V$.

- dim/rank formula: V, W finite dim., $\varphi\in\mathrm{Hom}(V,W) \Rightarrow \dim V = \dim\ker\varphi + \underbrace{\dim \mathrm{Im}\,\varphi}_{=\mathrm{rank}(\varphi)}$.

- $\exists$ bases (v_i) of V, (w_j) of W st. $M(\varphi) = \left(\begin{array}{c|c} I_{r\times r} & 0 \\ \hline 0 & 0 \end{array}\right)$ $\begin{matrix}\}\mathrm{Im}\,\varphi \\ \underbrace{\quad}_{\ker\varphi}\end{matrix}$

Lec. 9 - Dual: $V^* = \mathrm{Hom}(V,k)$.

(e_i) basis of V (finite dim) $\Rightarrow$ dual basis (e_i^*) of V^* st. $e_i^*(e_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$.

The annihilator of $U\subset V$ is $\mathrm{Ann}(U) = \{\ell\in V^* \mid \ell(u) = 0\ \forall u\in U\}$: $\dim \mathrm{Ann}(U) = n - \dim U$.

The transpose of $\varphi\in\mathrm{Hom}(V,W)$ is $\varphi^t: W^*\to V^*$, $\varphi^t(\ell) = \ell\circ\varphi$

- Quotient: $U\subset V$ subspace $\Rightarrow V/U = \{\text{cosets } v+U\}$ is a vector space.

  $V \xrightarrow{q} V/U$, $v\mapsto v+U$ is surjective with kernel $= U$.

  $V \xrightarrow{\varphi} W$ factors through V/U iff $U\subset\ker\varphi$.
  $q\searrow \quad \nearrow \exists\bar{\varphi}$ V/U

Linear operators:

Axler ch. 5 / Lec. 10

- $W \subset V$ is an *invariant subspace* for $\varphi \in \mathrm{Hom}(V,V)$ if $\varphi(W) \subset W$.
  Ex. $\ker(\varphi)$, $\mathrm{Im}(\varphi)$; *eigenspaces* $\mathrm{Ker}(\varphi - \lambda I)$.
- if $V = \oplus V_i$, V_i invariant for $\varphi \Rightarrow \exists$ basis where $M(\varphi)$ = block diagonal $\left(\begin{array}{c|c} \varphi|_{V_1} & 0 \\ \hline 0 & \varphi|_{V_2} \end{array}\right)$
  A basis of *eigenvectors* $v_i \in V$, $v_i \neq 0$, $\varphi(v_i) = \lambda_i v_i \Leftrightarrow \varphi$ *diagonalizable* $M(\varphi,(v_i)) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.
- Eigenvectors of φ for distinct eigenvalues are linearly indep^t

Lec. 11

- If k is algebraically closed (eg. $\mathbb{C}$) then any linear op. $\varphi \in \mathrm{Hom}(V,V)$ has an eigenvector.
  Corollary: $\exists$ basis in which $M(\varphi)$ is upper triangular $\begin{pmatrix} * & & \\ & \ddots & \\ 0 & & * \end{pmatrix}$
  $\lambda \in k$ is an eigenvalue of $\varphi \Leftrightarrow (\varphi - \lambda)$ not invertible $\Leftrightarrow \lambda$ appears on diagonal in a triangular matrix representing φ.

Math 55a Midterm 1 Review Problems

This is a fairly arbitrary selection of problems from the textbook that may be good for checking your understanding of the concepts. I also recommend going over the homework assignments (not worrying about any problems that were marked with a * or particularly tricky or lengthy).

Axler chapter 2: 2A.12; 2B.8; 2C.8, 2C.15, 2C.19

Axler chapter 3: 3B.3, 3B.5, 3B.19, 3B.22, 3B.24, 3B.25, 3B.27; 3C.2, 3C.7; 3D.5, 3D.10, 3D.17; 3E.1, 3E.15; 3F.20, 3F.22, 3F.24, 3F.25
(Note: Axler uses the notation $\mathbb{F}$ for the field; "null" for "Ker" and "range" for "Im"; $\mathcal{L}(V,W)$ for $\mathrm{Hom}(V,W)$; V' for V^*, U^0 for $Ann(U)$.)

Axler chapter 5: 5A.7, 5A.13, 5A.24, 5A.33; 5B.10; 5C.10; 5D.3, 5D.4; 5E.3

Artin chapter 2: 4.5, 4.7, 4.9, 4.10; 5.1, 5.2; 6.3, 6.10; 8.6, 8.7, 8.9, 8.10; 10.2; 11.1, 11.4; 12.1, 12.2; M.2, M.9, M.10
(A couple of these are a bit more involved than what is intended for problems on the midterm.)

Artin chapter 3: 1.9; 3.1; 4.5; 5.3.

Artin chapter 4: 4.2, 4.6.