ALREADY ON MIDTERM 1

- A <u>field</u> $(k,+,\times)$ is a set with two operations: $(k,+)$ abelian group with identity 0, $(k^\times = k-\{0\},\times)$ abelian group with identity 1; distributive law $a(b+c)=ab+ac$.
  <u>Examples</u>: $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ (characteristic 0: $\underbrace{1+\dots+1}_{n} = n\cdot 1 \neq 0$), $\mathbb{F}_p = \mathbb{Z}/p$ p prime (char. $= p$).

Axler ch.1

- A <u>vector space</u> over k is a set V with addition $+: V\times V \to V$ $(V,+)$ abelian group, $0\in V$
  scalar mult. $k\times V \to V$ associative, distributive.
  <u>Ex</u>: k^n, $k[x]$, … <u>Subspace</u>: $W\subset V$ closed under $+,\times$.
- $\mathrm{span}(v_1,\dots,v_n) = \{\sum a_i v_i \mid a_i \in k\} \subset V$, say (v_i) span V if $\mathrm{span}(v_i)=V$.

Axler ch.2

Say (v_i) are linearly independent if $a_1v_1+\dots+a_nv_n=0 \Rightarrow a_i=0\ \forall i$.
<u>Basis</u> = linearly independent vectors which span V: $k^n \xrightarrow{\sim} V$ isomorphism.
$(a_i) \mapsto \sum a_i v_i$

- All bases of V have same cardinality $=$ <u>$\dim V$</u>
  Any linearly independent set can be completed to a basis.
- $\mathrm{Hom}(V,W)$ = linear maps $\varphi: V\to W$, $\varphi(u+v)=\varphi(u)+\varphi(v)$, $\varphi(\lambda u)=\lambda\varphi(u)$.
  This is a vector space.
- given bases $(v_j)_{j=1..n}$ of V, $(w_i)_{i=1..m}$ of W, represent $v=\sum x_j v_j \in V$ by column $X=\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix}$

Axler ch.3

and $\varphi\in\mathrm{Hom}(V,W)$ by matrix $A=(a_{ij})$ whose columns represent $\varphi(v_j)$ in basis (w_i), $\varphi(v_j)=\sum_i a_{ij}w_i$.
Then $\varphi(v)$ is represented in basis (w_i) by column vector $Y=AX$.

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \text{basis}\uparrow\simeq & & \uparrow\simeq\text{basis} \\ k^n & \xrightarrow{A} & k^m \end{array}$$

Change of basis: $P=(p_{ij})=M(\mathrm{id},(v_i'),(v_i))$ ie. $v_j' = \sum p_{ij} v_i$,
then for $\varphi: V\to V$, $M(\varphi,(v_i'))=A'=P^{-1}AP$

- $V \simeq W_1\oplus\dots\oplus W_n$ direct sum decomp^n if $\begin{cases} W_i \text{ span } V: \forall v\in V\ \exists w_i\in W_i \text{ st. } v=w_1+\dots+w_n \\ W_i \text{ independent}: w_1+\dots+w_n=0,\ w_i\in W_i \Rightarrow w_i=0\ \forall i. \end{cases}$
  ie. $\varphi: \oplus W_i \xrightarrow{\sim} V$ is an isomorphism.
  $(w_i) \mapsto \sum w_i$
- V finite dim. $\Rightarrow V=W_1\oplus W_2$ iff $W_1\cap W_2=\{0\}$ and $\dim W_1+\dim W_2=\dim V$.
- <u>dim/rank formula</u>: V,W finite dim., $\varphi\in\mathrm{Hom}(V,W) \Rightarrow \dim V = \dim\ker\varphi + \underbrace{\dim \mathrm{Im}\,\varphi}_{=\mathrm{rank}(\varphi)}$.
- $\exists$ bases (v_i) of V, (w_j) of W st. $M(\varphi)=\left(\begin{array}{c|c} I_{r\times r} & 0 \\ \hline 0 & 0 \end{array}\right)\begin{matrix}\}\mathrm{Im}\,\varphi \\ \underbrace{\quad}_{\ker\varphi}\end{matrix}$

- <u>Dual</u>: $V^* = \mathrm{Hom}(V,k)$.
  (e_i) basis of V (finite dim) $\Rightarrow$ dual basis (e_i^*) of V^* st. $e_i^*(e_j)=\delta_{ij}=\begin{cases}1 & i=j\\ 0 & \text{else}\end{cases}$.
  $V \to V^{**}$
  $v \mapsto ev_v: \left(\begin{array}{c} V^*\to k \\ \ell\mapsto \ell(v)\end{array}\right)$ is an isomorphism if $\dim V<\infty$ (injective if $\dim V=\infty$).
  The annihilator of $U\subset V$ is $\mathrm{Ann}(U)=\{\ell\in V^* \mid \ell(u)=0\ \forall u\in U\}$; $\dim\mathrm{Ann}(U)=n-\dim U$.
  The <u>transpose</u> of $\varphi\in\mathrm{Hom}(V,W)$ is $\varphi^t: W^*\to V^*$, $\varphi^t(\ell)=\ell\circ\varphi$
  $\ker\varphi^t=\mathrm{Ann}(\mathrm{Im}\,\varphi)$, $\mathrm{Im}\,\varphi^t=\mathrm{Ann}(\ker\varphi)$ if $\dim<\infty$, $M(\varphi^t,(f_j^*),(e_i^*))=M(\varphi)^T$

- Quotient: $U \subset V$ subspace $\Rightarrow V/U = \{\text{cosets } v+U\}$ is a vector space.
  $V \xrightarrow{q} V/U$, $v \mapsto v+U$ is surjective with kernel $= U$.
  $V \xrightarrow{\varphi} W$ factors through V/U ($q \searrow V/U \dashrightarrow \exists \bar{\varphi}$) iff $U \subset \ker\varphi$.

ALREADY ON MIDTERM 1

Axler ch 5

- $W \subset V$ is an invariant subspace for $\varphi \in \mathrm{Hom}(V,V)$ if $\varphi(W) \subset W$.
  Ex. $\ker(\varphi)$, $\mathrm{Im}(\varphi)$; eigenspaces $\ker(\varphi - \lambda I)$.
- if $V = \oplus V_i$, V_i invariant for $\varphi \Rightarrow \exists$ basis where $M(\varphi) = \left(\begin{array}{c|c} \varphi|_{V_1} & 0 \\ \hline 0 & \varphi|_{V_2} \end{array}\right)$ block diagonal
  A basis of eigenvectors $v_i \in V$, $v_i \neq 0$, $\varphi(v_i) = \lambda_i v_i \Leftrightarrow \varphi$ diagonalizable, $M(\varphi,(v_i)) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.
- Eigenvectors of φ for distinct eigenvalues are linearly indep^t
- If k is algebraically closed (eg. $\mathbb{C}$) then any linear op. $\varphi \in \mathrm{Hom}(V,V)$ has an eigenvector.
  Corollary: $\exists$ basis in which $M(\varphi)$ is upper triangular $\begin{pmatrix} * & & \\ & \ddots & \\ 0 & & * \end{pmatrix}$
  $\lambda \in k$ is an eigenvalue of $\varphi \Leftrightarrow (\varphi - \lambda)$ not invertible $\Leftrightarrow \lambda$ appears on diagonal in a triangular matrix representing φ.

Axler ch 8

- The generalized kernel $g\ker(\varphi) = \ker(\varphi^N)$ $\forall N$ large (eg. $\geq \dim V$).
  φ is nilpotent if $\varphi^N = 0$; $\ker(\varphi) \subset \ker(\varphi^2) \subset \dots$ $\exists$ basis st. $M(\varphi)$ block diagonal with blocks $\begin{pmatrix} 0 & 1 & \\ & \ddots & 1 \\ 0 & & 0 \end{pmatrix}$
- generalized eigenspaces $V_\lambda = g\ker(\varphi - \lambda) = \ker(\varphi - \lambda)^N$ are linearly independent invariant subspaces.
- if k is alg. closed then $V =$ direct sum $\oplus V_\lambda$ of the gen^d eigenspaces of φ.
  This gives the Jordan normal form: $M(\varphi)$ block diagonal with blocks $\begin{pmatrix} \lambda & 1 & \\ & \ddots & 1 \\ 0 & & \lambda \end{pmatrix}$
  (φ diagonalizable $\Leftrightarrow$ all blocks have size 1).
- characteristic polynomial of φ: $\chi_\varphi(x) = \det(xI - \varphi) = \prod_{\lambda_i} (x-\lambda_i)^{n_i}$, $n_i = \mathrm{mult}(\lambda_i) = \dim V_{\lambda_i}$.
  minimal polynomial: $\mu_\varphi(x) = \prod (x-\lambda_i)^{m_i}$, $m_i = \min\{m \mid V_{\lambda_i} = \ker(\varphi-\lambda_i)^m\}$ = size of largest Jordan block in V_{λ_i}.
  - $p(\varphi) = 0$ iff $\mu_\varphi \mid p(x)$. In particular $\mu_\varphi \mid \chi_\varphi$.
    φ diagonalizable $\Leftrightarrow m_i = 1$ $\forall i$.
- Over $\mathbb{R}$, $\varphi: V \to V$ need not have eigenvectors, but by considering $V_{\mathbb{C}} = V \times V = \{v + iw \mid v, w \in V\}$ and $\varphi_{\mathbb{C}}: V_{\mathbb{C}} \to V_{\mathbb{C}}$, $\varphi_{\mathbb{C}}(v+iw) = \varphi(v) + i\varphi(w)$ $\Rightarrow$ any real operator has an invariant subspace of dimension 1 (eigenvector!) or 2.

Handout

NOT ON MIDTERM

- Categories have objects, and morphisms $\mathrm{Mor}(A,B)$ $\forall A, B \in \mathrm{ob}\,\mathcal{C}$, with operation = composition.
  Axioms: $\forall A \in \mathrm{ob}\,\mathcal{C}$, $\exists\, id_A \in \mathrm{Mor}(A,A)$, $f \circ id_A = id_B \circ f = f$; associativity $(f \circ g) \circ h = f \circ (g \circ h)$.
  Ex: sets, groups, vector spaces/k
- A functor $F: \mathcal{C} \to \mathcal{D}$ assigns
  - to each $X \in \mathrm{ob}\,\mathcal{C}$, $F(X) \in \mathrm{ob}\,\mathcal{D}$
  - to $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$, $F(f) \in \mathrm{Mor}_{\mathcal{D}}(F(X), F(Y))$

  st. $F(id_X) = id_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$. (contravariant functors = reverse dir^n of morphisms)
- Natural transformation t between functors $F, G: \mathcal{C} \to \mathcal{D}$
  = for each $X \in \mathrm{ob}\,\mathcal{C}$, $t_X \in \mathrm{Mor}_{\mathcal{D}}(F(X), G(X))$ st. $\forall$ $\begin{array}{ccc} X & & F(X) & \xrightarrow{t_X} & G(X) \\ \downarrow f & , & F(f) \downarrow & & \downarrow G(f) \\ Y & & F(Y) & \xrightarrow{t_Y} & G(Y) \end{array}$ commutes.

Axler ch 9A

- A bilinear form on V is $b: V \times V \to k$, linear in each input: $b(u+v,w) = b(u,w)+b(v,w)$, $b(\lambda u, v) = \lambda b(u,v)$ etc.
  b is symmetric if $b(u,v) = b(v,u)$, skew-symmetric if $b(u,v) = -b(v,u)$.
- $B(V) = \{\text{bilinear } b: V\times V \to k\} \xrightarrow{\sim} \mathrm{Hom}(V, V^*)$ (isom. of vector spaces)
  $b \longmapsto \varphi_b : v \longmapsto (b(v,\cdot): V \to k)$
  $\mathrm{rank}(b) = \mathrm{rank}(\varphi_b)$, b is nondegenerate if $\varphi_b : V \xrightarrow{\sim} V^*$ isomorphism.
- in a basis (e_i) of V, b is represented by a matrix $B = (b_{ij}) = (b(e_i, e_j))$.
  if $u = \sum x_i e_i$, $v = \sum y_i e_i$ are represented by column vectors X, Y, $b(u,v) = X^T B Y$.
- the orthogonal of $S \subset V$ for b is $S^\perp = \{v \in V / b(v,w) = 0 \ \forall w \in S\} = \mathrm{Ker}(V \to S^*)$, $v \mapsto \varphi_b(v)|_S$
  If b is nondegenerate then $\dim S^\perp = \dim V - \dim S$

Axler ch.6

  If b is an inner product then $S \cap S^\perp = \{0\}$ and $V = S \oplus S^\perp$.

- A real inner product $\langle \cdot,\cdot \rangle : V \times V \to \mathbb{R}$ is a symmetric definite positive bilinear form.
  Cauchy-Schwarz ineq: $\langle u, v\rangle \leq \|u\|\,\|v\|$. ↳ $\langle u,u\rangle = \|u\|^2 > 0 \ \forall u \neq 0$.

Over $\mathbb{C}$, we consider Hermitian inner products $\langle \cdot,\cdot \rangle : V \times V \to \mathbb{C}$, not quite bilinear: $\langle \lambda u, v\rangle = \bar\lambda \langle u, v\rangle$
require Hermitian-symmetric $\langle v, u\rangle = \overline{\langle u, v\rangle}$, and definite positive $\langle u,u\rangle = \|u\|^2 > 0 \ \forall u \neq 0$.
The map $V \to V^*$ induced by such $\langle\cdot,\cdot\rangle$ is $\mathbb{C}$-antilinear: $\varphi(\lambda u) = \bar\lambda \varphi(u)$.

- Every finite dimensional inner product space (over $\mathbb{R}$ or $\mathbb{C}$) has an orthonormal basis $(e_1, \dots, e_n)$ st. $\langle e_i, e_j\rangle = \delta_{ij}$. (build by induction eg. using Gram-Schmidt).

Axler ch.7

- Let $V, \langle\cdot,\cdot\rangle$ inner product space (over $\mathbb{R}$ or $\mathbb{C}$), $T: V \to V$ linear operator.
  The adjoint operator $T^*: V \to V$ satisfies $\langle v, Tw\rangle = \langle T^* v, w\rangle \ \forall v, w \in V$.
  (corresponds to the transpose of T via $V \xrightarrow[\varphi]{\sim} V^*$; over $\mathbb{C}$: complex conjugate of T^t).
  In an orthonormal basis, $M(T^*) = M(T)^t$ (real case) or $\overline{M(T)}^t$ (complex Hermitian case)
  $\mathrm{Ker}(T^*) = \mathrm{Im}(T)^\perp$ and vice-versa.
- $T: V \to V$ is self-adjoint if $T^* = T$
  T is orthogonal (unitary over $\mathbb{C}$) if $T^* = T^{-1}$, ie. $\langle Tu, Tv\rangle = \langle u, v\rangle \ \forall u, v \in V$.
  ($\Leftrightarrow$ T maps orthonormal basis to orthonormal basis)
- If $S \subset V$ is invariant under a self-adjoint/orthogonal/unitary operator then so is $S^\perp$.

$\Rightarrow$ Spectral theorem (real and complex versions):

- If $T: V \to V$ is self adjoint then T is diagonalizable, with real eigenvalues, and can be diagonalized in an orthonormal basis.
- If $T: V \to V$ is orthogonal for a real inner product, then V is a direct sum of orthogonal invariant subspaces of dim. 1 or 2, with T acting by ± 1 on the 1-dim! pieces, rotations on 2-dim! pieces.
- If $T: V \to V$ is unitary for a Hermitian inner product, then T is diagonalizable in an orthonormal basis, with eigenvalues $|\lambda_i| = 1$.
  More generally if $T: V \to V$ (Hermitian) is normal ie. $TT^* = T^*T \Rightarrow T$ is diagonalizable in an orthonormal basis.

- Besides inner products, one can also consider arbitrary nondegenerate symmetric bilinear forms (without assuming positivity); eg. over $\mathbb{R}$ (resp. $\mathbb{C}$), $\exists$ orthogonal basis st. $b(e_i,e_j) = \begin{cases} \pm 1 & i=j \\ 0 & i\neq j \end{cases}$ (resp. $b(e_i,e_j)=\delta_{ij}$); or skew-symmetric bilinear forms.

Handout

- Tensor product: $V\otimes W$ vector space, with a bilinear map $V\times W \to V\otimes W$, $(v,w)\mapsto v\otimes w$, st. bilinear maps $V\times W \xrightarrow{b} U$ correspond to linear maps $V\otimes W \xrightarrow{\varphi} U$ $(\varphi(v\otimes w)=b(v,w))$

  Elements of $V\otimes W$ are finite linear combinations $\sum v_i\otimes w_i$

  If (e_i) basis of V and (f_j) basis of W, then $(e_i\otimes f_j)$ basis of $V\otimes W$.

- $V^*\otimes W \simeq \mathrm{Hom}(V,W)$, by mapping $\ell\otimes w \in V^*\otimes W$ to $(v\mapsto \ell(v)w)\in \mathrm{Hom}(V,W)$.

- the trace $\mathrm{tr}(T:V\to V)=\sum\lambda_i\in k$ can be defined by $\mathrm{Hom}(V,V)\simeq V^*\otimes V \to k$, $\ell\otimes v\mapsto \ell(v)$

NOT ON EXAM

- multilinear maps $V_1\times\dots\times V_n\to U \Leftrightarrow$ linear maps $V_1\otimes\dots\otimes V_n\to U$.
- $V^{\otimes n} = V\otimes\dots\otimes V$ contains subspaces
  $\mathrm{Sym}^n(V)$ = symmetric tensors ($\leftrightarrow$ symmetric multilinear maps) $v_{\sigma(1)}\dots v_{\sigma(n)} = v_1\dots v_n$
  $\Lambda^n(V)$ = exterior powers: alternating tensors $v_{\sigma(1)}\wedge\dots\wedge v_{\sigma(n)} = (-1)^\sigma v_1\wedge\dots\wedge v_n$.
- if $\dim V = n$ then $\Lambda^n V$ has dim 1; for $T:V\to V$, $\Lambda^n T:\Lambda^n V\to\Lambda^n V$ is multiplication by a scalar, the determinant $\det(T)\in k$.

- The theory of modules over a ring $(R,+,\times)$ (elements need not have multiplicative inverses) is more complicated than that of vector spaces.

  Finitely generated modules need not have a basis; those that do are called free.

- $\mathbb{Z}$-modules $\Leftrightarrow$ abelian groups.

  Every finitely generated $\mathbb{Z}$-module M with generators $(e_1..e_n)$ is a quotient of $\mathbb{Z}^n$ $\left(\varphi:\begin{array}{l}\mathbb{Z}^n\twoheadrightarrow M\\ (a_i)\mapsto\sum a_i e_i\end{array}\right)$ and $\ker(\varphi)\subset\mathbb{Z}^n$ is itself a free module, i.e. $\exists\, T:\mathbb{Z}^m\to\mathbb{Z}^n$ st. $M\simeq \mathbb{Z}^n/\mathrm{Im}\,T$

(parts of Artin ch. 14)

→ via linear algebra over $\mathbb{Z}$, one finds:

Every finitely generated abelian group is $\simeq \mathbb{Z}^r\times\mathbb{Z}/n_1\times\dots\times\mathbb{Z}/n_k$ for some $r, n_1..n_k$.

(recall $\mathbb{Z}/m\times\mathbb{Z}/n\simeq\mathbb{Z}/mn$ if $\gcd(m,n)=1$)

Artin ch 2

- A group $(G,\cdot)$ is a set with an operation $\cdot: G\times G\to G$ st. (1) $\exists e\in G$ identity s.t. $eg=ge=g\ \forall g\in G$, (2) $\forall g\in G\ \exists$ inverse $g^{-1}\in G$ st. $gg^{-1}=g^{-1}g=e$, (3) associativity $(ab)c=a(bc)\ \forall a,b,c\in G$.
- A group is abelian if $\cdot$ is commutative ($ab=ba\ \forall a,b\in G$)
- Ex: $(\mathbb{Z},+)$, $(\mathbb{Z}/n,+)$, $(\mathbb{C}^*,\times)$, symmetric group S_n; $GL_n(\mathbb{R})$ etc.; products $G\times H$, $\mathbb{Z}^n$, ...
- Like sets, groups can be finite ($\mathbb{Z}/n$, S_n,...), countable ($\mathbb{Z}$, $\mathbb{Z}^n$, $\mathbb{Q}$,...), uncountable ($\mathbb{R}$...)
- $H\subset G$ is a subgroup if $e\in H$, $a\in H\Rightarrow a^{-1}\in H$, $a,b\in H\Rightarrow ab\in H$. $|H|$ divides $|G|$.
  H,H' subgroups of $G\Rightarrow H\cap H'$ is a subgroup of G.
  All subgroups of $(\mathbb{Z},+)$ are $\mathbb{Z}n=\{mn/m\in\mathbb{Z}\}$ for some $n\geq 0$.
- A homomorphism $\varphi: G\to H$ is a map st. $\varphi(ab)=\varphi(a)\varphi(b)\ \forall a,b\in G$. ($\Rightarrow\varphi(a^{-1})=\varphi(a)^{-1}$)
  isomorphism = bijective homomorphism, automorphism = isom $G\xrightarrow{\sim}G$. $(Aut(G),\circ)$ is a group.
- The kernel of $\varphi: G\to H$: $\ker(\varphi)=\{g\in G/\varphi(g)=e_H\}$ subgroup of G. φ injective $\Leftrightarrow\ker\varphi=\{e\}$
  The image of φ: $Im(\varphi)=\{\varphi(g)/g\in G\}\subset H$ subgroup of H. φ surjective $\Leftrightarrow Im\,\varphi=H$.
- Given $a\in G$, $\varphi: \mathbb{Z}\to G$, $k\mapsto a^k$ is a homomorphism with $im(\varphi)=\langle a\rangle$ subgp generated by a.
  $\ker(\varphi)=\mathbb{Z}n$ where $n=$ order of $a=\min\{n>0$ st. $a^n=e\}$.
  Hence the cyclic group $\langle a\rangle$ is $\simeq\mathbb{Z}/n$ if a has order n, $\simeq\mathbb{Z}$ if infinite order.
  ($a_1,\dots,a_k$ generate G if every element of G is a product of a_i and their inverses).
- A subgroup $H\subset G$ determines an equivalence relation (axioms: $a\sim a$; $a\sim b\Leftrightarrow b\sim a$; $\left.\begin{matrix}a\sim b\\ b\sim c\end{matrix}\right\}\Rightarrow a\sim c$)
  $a\sim b$ iff $a^{-1}b\in H$, whose equivalence classes are the (left) cosets $aH=\{ah/h\in H\}$.
  The quotient set: $G/H=\{\text{cosets } aH\}$. The index of H: $(G:H)=|G/H|=\frac{|G|}{|H|}$ if finite.
- If G is finite: $H\subset G$ subgroup $\Rightarrow |H|$ divide $|G|$; $a\in G\Rightarrow ord(a)\,|\,|G|$; $|G|=p$ prime $\Rightarrow G\simeq\mathbb{Z}/p$.
- A subgroup $H\subset G$ is normal $\Leftrightarrow aH=Ha\ \forall a\in G\Leftrightarrow aHa^{-1}=H\ \forall a\in G$.
  (left cosets = right cosets) (conjugate subgroups)
- The operation $(aH)\cdot(bH)=abH$ makes G/H a group iff H is a normal subgroup.
- $\forall\varphi: G\to H$ homomorphism, $\ker(\varphi)=K$ is a normal subgroup of G, and $Im(\varphi)\simeq G/K$.
  If φ is surjective, we have an exact sequence $\{1\}\to K\xrightarrow[\text{injective}]{i}G\xrightarrow[\text{surjective}]{\varphi}H\to\{1\}$ $Im(i)=\ker(\varphi)$.
  Ex: $\{1\}\to H\xrightarrow[\text{normal}]{}G\to G/H\to\{1\}$; $0\to\mathbb{Z}/m\to\mathbb{Z}/mn\to\mathbb{Z}/n\to 0$ ($\mathbb{Z}/mn\simeq\mathbb{Z}/m\times\mathbb{Z}/n$ iff $\gcd(m,n)=1$)
  $\{e\}\to\mathbb{Z}/3\to S_3\xrightarrow{\text{sign}}\mathbb{Z}/2\to\{e\}$
  A homomorphism $G\xrightarrow{\varphi}H$ factors through $G\to G/K\xrightarrow{\bar\varphi}H$ iff $K\subset\ker\varphi$
- G is simple if its only normal subgroups are $\{e\}$ and itself. Ex: $\mathbb{Z}/p$ p prime; A_n $n\geq 5$.
- Ex: the center $Z(G)=\{z\in G/zg=gz\ \forall g\in G\}$ is a normal subgroup (abelian: $zz'=z'z$)
- Ex: the commutator subgroup $[G,G]=\{\prod_{\text{finite}}[a_i,b_i]\}$, where $[a,b]=aba^{-1}b^{-1}$, is normal, and $G/[G,G]=Ab(G)$ (abelianization) largest abelian quotient of G. $\forall G\xrightarrow{\varphi}H$, H abelian factors $G\to Ab(G)\xrightarrow{\bar\varphi}H$.
- Every finitely generated abelian group is $\simeq\mathbb{Z}^r\times\mathbb{Z}/n_1\times\dots\times\mathbb{Z}/n_k$ for some $r,n_1,\dots,n_k$.
  (recall $\mathbb{Z}/m\times\mathbb{Z}/n\simeq\mathbb{Z}/mn$ if $\gcd(m,n)=1$)

Artin 14.7

ALREADY ON MIDTERM 1

Artin 6.7–6.12

- **Group actions**: G-action on set S: $G \times S \to S$, $(g,s) \mapsto g.s$ s.t. $e.s = s$ $\forall s \in S$, $(gh).s = g.(h.s)$ ($\Leftrightarrow$ homom. $\rho: G \to \mathrm{Perm}(S)$) (6)

  faithful if ρ injective; transitive if $\forall s,t \in S$ $\exists g$ s.t. $g.s = t$ (ie: 1 orbit)

- The orbit of $s \in S$ is $\mathcal{O}_s = G.s = \{g.s \mid g \in G\}$. These form a partition $S = \coprod$ orbits.
  The stabilizer of s is $\mathrm{Stab}(s) = \{g \in G \mid g.s = s\}$ subgroup of G.
  Elements in same orbit have conjugate stabilizer subgroups $\mathrm{Stab}(g.s) = g\,\mathrm{Stab}(s)\,g^{-1} \subset G$.
  Orbit-stabilizer: if $H = \mathrm{Stab}(s)$, then $G/H \simeq \mathcal{O}_s$ bijection, in particular $|\mathcal{O}_s| \cdot |\mathrm{Stab}\, s| = |G|$.
  $gH \mapsto g.s$

- Burnside's lemma (G, S finite): let $S^g = \{s \in S \mid gs = s\}$ fixed points of $g \in G$, then $\#\text{orbits} = \frac{1}{|G|} \sum_{g \in G} |S^g|$

Artin ch. 7

- G acts on itself by left multiplication. This gives $G \hookrightarrow \mathrm{Perm}(G)$, hence:
  every finite group G is isomorphic to a subgroup of S_n, $n = |G|$.

- G acts on itself by conjugation: g acts by $h \mapsto ghg^{-1}$.
  orbits = conjugacy classes; $\mathrm{Stab}(h) = \{g \in G \mid gh = hg\} = Z(h)$ centralizer of h.
  Hence: $|G| = \sum_{C \subset G \text{ conj. classes}} |C|$, where for each conj. class $|C_h| = \frac{|G|}{|Z(h)|}$ divides $|G|$. (class eq^n of G)

- For p-groups ($|G| = p^k$), the class equation $\Rightarrow |Z(G)| \geq p$ (number of conj. classes of size 1)
  Hence: $|G| = p^2$, p prime $\Rightarrow$ G is abelian ($\simeq \mathbb{Z}/p \times \mathbb{Z}/p$ or $\mathbb{Z}/p^2$)
  - 5 isom. classes of groups of order 8: $\mathbb{Z}/8$, $\mathbb{Z}/4 \times \mathbb{Z}/2$, $(\mathbb{Z}/2)^3$, D_4, quaternion group.

- $G \subset SO(3)$ finite subgp $\Rightarrow$ by considering the action of G on its poles (unit vectors along rotation axes),
  $G \simeq$ one of $\mathbb{Z}/n$, D_n (regular n-gon), A_4 (tetrahedron), S_4 (cube), A_5 (dodecahedron/icosahedron)

- The symmetric group S_n is generated by transpositions (ij), in fact by $s_i = (i\ i+1)$.
- $\forall \sigma \in S_n$ $\exists$ unique decomp of σ as product of disjoint cycles $(a_1 \dots a_k)$.
  $\sigma, \tau \in S_n$ are in same conjugacy class iff they have the same cycle lengths.
- the alternating group $A_n = \ker(\mathrm{sign}: S_n \to \mathbb{Z}/2) = \{\text{products of even \# of transpositions}\}$
  A conjugacy class in S_n which consists of even permutations is either 1 or 2 conj. classes in A_n;
  it splits into 2 iff the centralizer $Z(\sigma) \subset A_n$ ($\Leftrightarrow$ cycle lengths of σ are all odd & distinct).
- A_n is simple for $n \geq 5$ (A_4 isn't: $\{id, (ij)(kl)\} \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ is normal in A_4 and S_4)

Math 55a Midterm Review Problems

This is a fairly arbitrary selection of problems from the textbook that may be good for checking your understanding of the concepts. I also recommend going over the homework assignments (not worrying about any problems that were particularly tricky or lengthy).

Problem numbers in Axler refer to the 4th edition.

Midterm 1

Axler chapter 2: 2A.12; 2B.8; 2C.8, 2C.15, 2C.19

Axler chapter 3: 3B.3, 3B.5, 3B.19, 3B.22, 3B.24, 3B.25, 3B.27; 3C.2, 3C.7; 3D.5, 3D.10, 3D.17; 3E.1, 3E.15; 3F.20, 3F.22, 3F.24, 3F.25
(Note: Axler uses the notation $\mathbb{F}$ for the field; "null" for "Ker" and "range" for "Im"; $\mathcal{L}(V, W)$ for $\mathrm{Hom}(V, W)$; V' for V^*, U^0 for $Ann(U)$.)

Axler chapter 5: 5A.7, 5A.13, 5A.24, 5A.33; 5B.20; 5C.10; 5D.3, 5D.4; 5E.3

Artin chapter 2: 4.5, 4.7, 4.9, 4.10; 5.1, 5.2; 6.3, 6.10; 8.6, 8.7, 8.9, 8.10; 10.2; 11.1, 11.4; 12.1, 12.2; M.2, M.9, M.10
(A couple of these are a bit more involved than what is intended for problems on the midterm.)

Artin chapter 3: 1.9; 3.1; 4.5; 5.3.

Artin chapter 4: 4.2, 4.6.

Midterm 2

(Problems marked * are a bit more involved than what is intended for the midterm.)

Axler chapter 6: 6A.2; 6B.6*, 6B.15, 6B.21*; 6C.7, 6C.10*, 6C.12
(Note for 6C: Axler denotes by P_U the orthogonal projection onto $U \subset V$, see 6.55)

Axler chapter 7: 7A.4, 7A.5, 7A.6, 7A.23, 7A.27 (assume self-adjoint rather than normal); 7B.10, 7B.14, 7B.21, 7B.22, 7B.23*; 7C.19; 7D.8

Axler chapter 8: 8A.2, 8A.22; 8B.2, 8B.9; 8C.4, 8C.7; 8D.1, 8D.2

Artin chapter 4: 4.2, 4.6; 6.4, 6.8; 7.3, 7.6, 7.7; M.2*, M.7*, M.10*

Artin chapter 5: 2.3*, M.3, M.7*

Artin chapter 6: 4.2*, 4.3; 7.1, 7.6, 7.10, 7.11*; 8.4; 9.2, 9.4; 10.2, 10.3*; 11.1, 11.3, 11.5, 11.7*; M.3, M.4*

Artin chapter 7: 1.2; 2.2, 2.3*, 2.7, 2.13*, 2.17*; 3.2, 3.3; 4.2, 4.9*; 5.2, 5.3, 5.7, 5.12; M.6, M.7.

Artin chapter 8: 2.2; 4.2, 4.3, 4.10, 4.12; 5.2, 5.3, 5.4*; 6.2, 6.19; M.4*, M.14*

Artin chapter 14: 7.2, 7.5, 7.9*