Proofwriting Basics for Math 55

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Introduction

The goal of a proof is to explain logically why a claim must be true, from previous results, in a logically infallable and rigorous way. Producing a rigorous proof confirms that a statement is true, as long as all the assumptions used have been proved as well. Proofs also illuminate the underlying reasons why exactly a certain pattern or condition always hold.

Proofwriting Techniques

A proof that follows a direct chain of reasoning is known as a direct proof. This is likely the most common type of proof you will use. In general, using a direct proof can be more illuminative and clear than the techniques described below, although they all have their place in rigorous mathematical proofs.

Claim 0.1. *The square of an even number is even.*

Proof. Let n be even. This means we can write $n = 2k$ for some $k \in \mathbb{N}$. Then, $n^2 = (2k)^2 = 4k^2$ is a multiple of four, and thus even. □

Proof By Contradiction

If you are trying to show that some statement is true, it works just as well to show that the alternative is false. Proof by contradiction involves taking the direct opposite of a statement, and showing that it logically implies

Claim 0.2. *The square root of two is irrational.*

Proof. For the sake of contradiction, let the square root of two be rational. Then, there exist coprime $p, q \in \mathbb{Z}$ such that $\sqrt{2} = \frac{p}{q}$, or equivalently, $2q^2 = p^2$. This means p^2 is even, so p must also be even. We can write $p = 2r$, for some $r \in \mathbb{Z}$. Then, $2q^2 = (2r)^2 = 4r^2$, so $q^2 = 2r^2$, meaning q^2 and thus q are even. However, p and q both being even contradicts them sharing no factors. □

Proof By Induction

Proof by Induction is a strategy that is often compared to climbing up the rungs of the latter. First, you prove the statement for the base case, typically $n = 1$ or $n = 2$. Then, you assume the

inductive hypothesis, which is that the statement holds up to some n, and with that prove it works for $n+1$ (the inductive case). Together, these two cases prove that the statement holds for all n. You can either use induction on a fact that is directly about the natural numbers, or on objects which can be labeled by a natural number. For example, you can prove facts about a group by inducting on the number of elements, or prove facts about graphs by inducting on the number of vertices.

Claim 0.3. $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Proof. We proceed by induction on n. For $n = 1$, observe that $1 = \frac{1 \cdot 2}{2}$. Now, if $1+2+\cdots+n = \frac{n(n+1)}{2}$ for some n, then

$$1 + 2 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + n + 1 = \frac{n^2 + n + 2n + 2}{2} = \frac{(n+1)(n+2)}{2},$$

which concludes the inductive case. □

Proof By Contrapositive

The contrapositive of the statement "if A then B" is "if not B, then not A." This is logically equivalent to the original claim (why?). Make sure not to confuse the contrapositive with the converse, which is "if B, then A."

Claim 0.4. *If ab is odd for integers a, b, then both a, b are odd.*

Proof. We shall prove that if one of a, b is even, then ab is even. Without loss of generality, let a be even. This means we can write $a = 2c$, for some integer c. Then, $ab = 2bc$ is even. □

Common Proofwriting Errors

These are some common errors that can often come up when beginning your proofwriting journey.

- **Lack of Rigor**

  Every step of your proof should be justified, either by a definition, a previous result, or valid manipulation. This makes sure that there are no flaws in the argument, and helps make your line of reasoning more clear.

- **Handwaving**

  Saying "it can be shown" or "one can check" may make sense for explaining a line of reasoning to a friend, but in a written proof, you need to actually show or check your claim. Make sure to explicitly show algebraic manipulations and casework. Also, use formal induction rather than ellipses to show an arguement continuing.

- **Lack of Specificity**

  Make sure to label variables ("there exists such an x") rather than referring to "some number." Also, name the theorems you use ("by the first isomorphism theorem"). This does not have to be done for every single thing you reference, but it is better to err on the side of more specificity.

- **Overcomplication**

  If there are two valid ways of proving a claim, the shorter one is the better proof. You won't lose points in 55 for making a longer proof, but it is good practice to try to make your arguments simpler. This often involves using strong (very applicable and powerful) theorems rather than brute calculations or starting from first principles. Simplifying proofs can also be a massive timesaver for you.

Tips for Proofwriting in 55

- **Utilize Course Notes**

  Luckily for you, Math 55 is a very self-contained course, so there is very little you will have to reference from outside of what is explained in lecture. Use this to your favor. If you are unsure how to approach a problem about a certain concept, look back at the Lecture Notes or your own personal notes on that concept. Chances are, you'll find the key theorem or fact for your proof.

- **Start With Definitions**

  It's very difficult to prove a statement that you don't understand. Making sure you understand the definitions of everything involved with the problem makes sure you are equipped to tackle it. Also, since the conclusion of a proof must follow directly from definitions and previous theorems, figuring out what the definitions imply gets you halfway towards solving the problem.

- **Prioritize Clarity**

  You will be writing complex proofs that require many steps, so you should always make sure that you fully understand every step, and that your explanation fully conveys your thoughts. If your solution couldn't fully convince someone else that your reasoning is correct if read aloud, it probably needs to be more clear.

- **Take a Walk**

  Many of the problems you will see cannot be solved in a single go-through. Make sure to take a step back from a problem if you have been dedicating too much time to it, and let yourself come back to it with a clear mind.

- **Collaborate!**

  Working with other people is massively helpful for tackling problems, and learning how to communicate your mathematical ideas aloud also makes writing proofs easier.