Characters

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1 definition and properties

Throughout this handout, we assume that G is a finite group and all representations are finite-dimensional and complex.

Definition 1.1. Let V be a representation of G. The *character* $\chi_V : G \to \mathbb{C}$ is the map $g \mapsto \mathrm{tr}(g : V \to V)$.

In general, χ_V has no reason to be a homomorphism. However, it takes the same value on conjugate elements of G because the trace of a linear operator is invariant under conjugation. For this reason, if we let $\mathcal{C}$ denote the set of conjugacy classes of G, then χ_V can be viewed as a map belonging to the space $\{\text{functions } \mathcal{C} \to \mathbb{C}\} = \mathbb{C}^{\mathcal{C}}$. We call χ_V a *class function*.

Proposition 1.1. For representations V, W of G, the character has the following properties:

(1) $\chi_V(e) = \dim(V)$

(2) $\chi_{V\oplus W} = \chi_V + \chi_W$

(3) $\chi_{V\otimes W} = \chi_V \cdot \chi_W$

(4) $\chi_{V^*} = \overline{\chi_V}$

(5) $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V(g)}\chi_W(g)$

(6) $\chi_{Sym^2(V)}(g) = \frac{\chi_V(g)^2+\chi_V(g^2)}{2}$

(7) $\chi_{\wedge^2 V}(g) = \frac{\chi_V(g)^2-\chi_V(g^2)}{2}$

Proof. (1): $\chi_V(e) = \mathrm{tr}(\mathrm{id}_V) = \dim V$.

(2)-(4): Fix $g \in G$. Fix an eigenbasis of $\{v_i\}$ of V for $g : V \to V$, and fix an eigenbasis $\{w_j\}$ of W for $g : W \to W$; let $\{\lambda_i\}$ and $\{\mu_j\}$ denote the corresponding eigenvalues. The direct sum $V \oplus W$ has an eigenbasis $\{v_i\} \cup \{w_j\}$ with eigenvalues preserved, and $V \otimes W$ has an eigenbasis $\{v_i \otimes w_j\}$ with eigenvalues $\{\lambda_i\mu_j\}$. The dual V^* has an eigenbasis $\{v_i^*\}$ with eigenvalues $\lambda_i^{-1} = \overline{\lambda_i}$ (remember the λ_i are unit complex numbers).

(5) is a consequence of (3), (4), and the isomorphism $\mathrm{Hom}(V, W) \cong V^* \otimes W$. Use suitable eigenbases for $\mathrm{Sym}^2(V)$ and $\wedge^2(V)(g)$ to work out (6) and (7). □

2 projection operator, Hermitian inner product, and consequences

A representation V of G contains some number of copies of the one-dimensional trivial representation V_0. The subspace $V_0^{\oplus a_0} \subseteq V$ can be characterized as the subspace $V^G \subseteq V$ consisting of the vectors $v \in V$ fixed under G; that is, the vectors v such that that $gv = v$ for all $g \in G$. We showed that the operator

$$\varphi = \frac{1}{|G|} \sum_{g \in G} g : V \to V$$

is a homomorphism of representations and a projection onto V^G. As a consequence,

$$\dim V^G = tr(\varphi) = \frac{1}{|G|} \sum_{g \in G} tr(g) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g).$$

Now, for representations V, W of G, we can apply the above equality to the representation $\mathrm{Hom}(V, W)$. On your homework, you verified that $\mathrm{Hom}(V, W)^G$ is exactly the subspace $\mathrm{Hom}_G(V, W) \subseteq \mathrm{Hom}(V, W)$ of G-equivariant maps $V \to W$. We have

$$\dim \mathrm{Hom}_G(V, W) = \dim \mathrm{Hom}(V, W)^G = \frac{1}{|G|} \sum_{g \in G} \chi_{\mathrm{Hom}(V,W)}(g) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \chi_W(g).$$

To understand the important consequences of this result, we define an Hermitian inner product on the space $\mathbb{C}^{\mathcal{C}}$ of class functions on G by

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \cdot \beta(g).$$

In terms of this Hermitian inner product, we proved for representations V, W of G that

$$\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_G(V, W).$$

If V and W are irreducible, then Schur's Lemma implies

$$\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the characters of the irreducible representations are orthonormal with respect to our Hermitian inner product on $\mathbb{C}^{\mathcal{C}}$. Orthonormality implies linear independence, bounding the number of irreps of G by the number of conjugacy classes of G.

We will soon prove that

Theorem 2.1. The characters of the irreducible representations of G span the space of class functions $\mathbb{C}^{\mathcal{C}}$ on G.

The important implication is that the characters of the irreps form not just an orthonormal set but an *orthonormal basis* for the space of class functions on G. As a result,

Corollary 2.2. The number of irreducible representations of G is equal to the number of conjugacy classes of G.

Denote by $V_0, V_1, \ldots, V_k$ the distinct irreps of G where V_0 is the one-dimensional trivial representation. Any representation V has a decomposition $V = V_0^{\oplus a_0} \oplus \cdots \oplus V_k^{\oplus a_k}$.

Corollary 2.3. A representation is determined by its character. As seen earlier,

$$\langle \chi_V, \chi_{V_i} \rangle = \dim \mathrm{Hom}_G(V, V_i)$$

returns a_i, the number of copies of V_i in the decomposition of V into irreducibles. Notice that $\dim V^G = \dim V_0^{\oplus a_0} = \langle \chi_V, \chi_{V_0} \rangle$.

Corollary 2.4. Observe that

$$\dim \mathrm{Hom}_G(V, V) = \langle \chi_V, \chi_V \rangle = \sum_{i,j=0}^{k} a_i a_j \langle \chi_{V_i}, \chi_{V_j} \rangle = \sum_{i=0}^{k} a_i^2 \langle \chi_{V_i}, \chi_{V_i} \rangle = \sum_{i=0}^{k} a_i^2.$$

In particular, V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.

3 permutation representations

Permutation representations are those associated to an action $G \to \mathrm{Perm}(S)$ of a finite group G on a finite set S. One defines an $|S|$-dimensional complex vector space V: start with a basis $\{e_s\}_{s \in S}$, and set $V := \{\sum_{s \in S} c_s e_s \mid c_s \in \mathbb{C}\}$. In other words, we define V to be the vector space of functions $S \to \mathbb{C}$.

An element of $g \in G$ acts linearly on V via the mapping $e_s \mapsto e_{gs}$, and this action of G gives V the structure of a G-representation. In the given basis, each $g : V \to V$ is represented by a permutation matrix. The character $\chi_V(g)$ is the number of 1s along the diagonal of the associated permutation matrix, i.e. the number of the basis vectors e_s fixed by g, i.e. the number of fixed points $|S^g|$.

3.1 the regular representation

We've studied some special cases of this construction, one being the *regular representation* R associated to the action of G on itself by left multiplication (concretely, $g(e_h) = e_{gh}$ for elements $g, h \in G$). The regular representation of G is faithful, which implies that any every finite abstract group can be embedded in a linear group GL_n where $n = |G|$. (This is analogous to how the existence of this faithful group action $G \to \mathrm{Perm}(G)$ implies that every finite group can be realized as a subgroup of S_n.)

The character of the regular representation is

$$\chi_R(g) = \begin{cases} |G| & \text{if } g = e \\ 0 & \text{else.} \end{cases}.$$

Let $V_1, \ldots, V_k$ denote the irreps of G. From the calculation

$$\langle \chi_R, \chi_{V_i} \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_R(g)} \chi_{V_i}(g) = \frac{1}{|G|} \left(\overline{\chi_R(e)} \chi_{V_i}(e) \right) = \frac{1}{|G|} (|G| \cdot \dim V_i) = \dim V_i,$$

we see that the decomposition

$$R = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$$

of R as a direct sum of the V_i has $a_i = \dim V_i$. Notice that every irreducible representation of G appears as a subrepresentation of the regular representation.

We deduce from this decomposition that the dimensions of the irreducible representations of G are controlled by the size of the group. From direct calculation, we have

$$\langle \chi_R, \chi_R \rangle = \frac{1}{|G|} \left(\overline{\chi_R(e)} \chi_R(e) \right) = \frac{|G|^2}{|G|} = |G|,$$

while by Corollary 2.4,

$$\langle \chi_R, \chi_R \rangle = \sum a_i^2 = \sum (\dim\, V_i)^2.$$

Thus, if $V_1, \ldots, V_k$ are the irreducible representations of G, then

$$|G| = \sum_{i=1}^{k} (\dim\, V_i)^2.$$

3.2 the standard representation

The symmetric group S_n acts on $\mathbb{C}^n$ by permutations of the standard basis $\{e_i\}$. Denote this n-dimensional permutation representation by P. The subspace

$$U = \{(z_1, \ldots, z_n) \mid z_1 = \cdots = z_n\} = \text{span}(e_1 + \cdots + e_n)$$

is invariant under this action; P contains a copy of the one-dimensional trivial representation.

The vectors in the complement $V := U^\perp$ are the $w = \sum a_i e_i$ with $\sum a_i = 0$. For any $\sigma \in S_n$ and $v = \sum a_i e_i \in V$, we have $\sum a_{\sigma(i)} = \sum a_i = 0$. The space V is invariant under S_n, and is therefore a representation of S_n in its own right. We call V the *standard representation* of S_n. Notice that V is spanned by the $n-1$ vectors $u_i = \{e_i - e_n\}_{1 \le i \le n-1}$, as any element w of V can be written as

$$w = a_1 e_1 + \cdots + a_{n-1} e_n - (a_1 + \cdots + a_{n-1}) e_n = \sum_{i=1}^{n-1} a_i u_i.$$

In fact, the u_i are linearly independent and thus form a basis for V.

Proposition 3.1. The standard representation is irreducible.

Proof. Let $v = \sum c_i e_i$ be any nonzero vector in V. Define $W \subseteq V$ to be the subrepresentation of V generated by v, meaning the smallest subrepresentation of V containing v. (Concretely, W is the span of the vectors $\{gv\}_{g \in G}$.) We show that $W = V$.

There must exist k, ℓ such that $c_k \neq c_\ell$. Letting $\sigma_{k\ell}$ denote the transposition in S_n swapping k and ℓ, we have

$$\frac{1}{c_k - c_\ell}(v - \sigma_{k\ell}(v)) = \frac{1}{c_k - c_\ell}(c_k - c_\ell)(e_k - e_\ell) = e_k - e_\ell.$$

For each $i \in \{1, \ldots, n-1\}$, the vector $\sigma_{(ik)}\sigma_{(\ell n)}(e_k - e_\ell) = u_i$ belongs to W. We are done because the u_i span V. □

Using the decomposition $P = U \oplus V$, we can express the character of V in terms of fixed points where $S = \{1, 2, \ldots, n\}$:

$$\chi_V(g) = \chi_P(g) - \chi_U(g) = |S^g| - 1.$$

4 Miscellaneous observations

- The first column of a character table lists $\chi_V(e) = \dim V$. Recall that $|G| = \sum(\dim V_i)^2 = \sum \chi_{V_i}(e)^2$.
- If V is a one-dimensional (hence irreducible) representation of G and $g \in G$ has order n, then $\chi_V(g)$ is an n-th root of unity.
- If there is only one irreducible representation W of dimension n, then $W^* \cong W$, which implies that $\chi_W = \overline{\chi_W}$ for all $g \in G$. Moreover, for any one-dimensional representation U, we have $W \otimes U \cong W$, and thus $\chi_W(g)\chi_U(g) = \chi_W(g)$ for all $g \in G$. The consequence we've been using is that $\chi_U(g) \neq 1$ implies $\chi_W(g) = 0$.
- If the character of a representation V is real-valued ($\chi_V(g) \in \mathbb{R}$ for all $g \in G$), then $V \cong V^*$. Hence, if χ_{V_i} is real-valued for every irreducible representation V_i, then every representation of G is isomorphic to its dual.

I won't write this up because tables are easier to draw on a board than to LaTeX, but an exercise I recommend is to take the standard representation V of S_4 find the decomposition of $V \otimes V$ into irreducibles. Answer in the footnotes.[1]

[1] In lecture notation, $V \otimes V$ is isomorphic to $U \oplus V \oplus V' \oplus W$.