Representation Theory: Finite Groups

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1 basics

1.1 definitions

Definition 1.1. Let G be a group. A representation of G is a vector space V along with a homomorphism $\rho : G \to GL(V)$.

In 55, we are concerned almost exclusively with finite groups and finite-dimensional vector spaces over $\mathbb{C}$. Complete reducibility, one of our most important statements, depends upon working over a characteristic zero field. We use algebraic closure to prove Schur's Lemma and the fact that every operator $g : V \to V$ is diagonalizable. Our approach can be somewhat modified to compensate for the loss of algebraic closure, but it completely breaks down without the characteristic zero assumption. Representation theory in characteristic p looks totally different. Another technicality: a few of my statements might only be accurate modulo the case $V = 0$.

If ρ sends all elements of G to the identity transformation in $GL(V)$, then it is called the *trivial representation* in dimension $n = \dim V$. At the opposite end, injective homomorphisms $G \to GL(V)$ are embeddings (*faithful* representations); with no loss of information, we can view G as a linear group. I was told that before mathematicians came up with the notion an abstract group, the groups they thought about were just sets of invertible matrices closed under inverses and matrix multiplication. The study of representation theory is more general—we want to describe all of the ways that a given abstract group G can be mapped to a linear group GL_n.

Definition 1.2. Let V and W be representations of G. A linear map $\phi : V \to W$ is a *homomorphism of representations* or a *G-equivariant linear map* if it respects the action of G, meaning that for all $g \in G$ and $v \in V$, we have $\phi(g_V \cdot v) = g_W \cdot \phi(v)$.

The set of G-equivariant linear maps $V \to W$, denoted $\mathrm{Hom}_G(V, W)$, forms a vector subspace of $\mathrm{Hom}(V, W)$.

Definition 1.3. If V is a representation of G, then a *subrepresentation* of V is a subspace $W \subseteq V$ invariant under the action of G, meaning $g(W) = W$ for all $g \in G$.

A subrepresentation is a representation of G in its own right, one that happens to be a subspace of V. We could have defined a subrepresentation of V equivalently as a representation W of G along with a G-equivariant injection $W \hookrightarrow V$ (e.g. the inclusion, if W is already realized as a subspace of V).

Irreducible representations are those that cannot be decomposed any further:

Definition 1.4. A representation V of G is *irreducible* if it has no subrepresentations except for $W = 0$ and $W = V$.

Because any subspace of a trivial representation is invariant under G, the only irreducible trivial representation is the trivial representation of dimension 1.

Lemma 1.1. If G is a finite group and V is finite-dimensional, complex representation of G, then each $\rho(g) : V \to V$ is diagonalizable, and the eigenvalues of g are roots of unity.

Proof. Observe that the operator $\rho(g) \in GL(V)$ is of finite order (its order divides the order of g). From here, use the minimal polynomial as we did in lecture. A different proof can be found in Artin under Theorem 4.7.14. □

1.2 constructions

Given representations V and W of a group G, we can turn $V \oplus W$ into a G-representation by defining $g(v, w) := (gv, gw)$. The actions on the V and W factors don't interact. Similarly, $V \otimes W$ can be given the structure of a G-representation by setting $g(v \otimes w) := gv \otimes gw$.

To define the dual of a representation $\rho : G \to GL(V)$, consider the natural bilinear pairing $V \times V^* \to \mathbb{C}$ given by $(v, \ell) \mapsto \ell(v)$. We want the action of G on V^* to respect this pairing, meaning $(g_V(v), g_{V^*}(\ell)) \mapsto \ell(v)$. So, we define $\rho^*(g) = {}^t\rho(g^{-1}) : V^* \to V^*$, which satisfies

$$g_{V^*}(\ell)(g_V(v)) = (\ell \circ g_V^{-1})(g_V(v)) = \ell(g_V^{-1} g_V(v)) = \ell(v).$$

Similarly, when we give $\mathrm{Hom}(V, W)$ the structure of a G-representation, we want the action to satisfy $g(\varphi)(gv) = \varphi(v)$ for any $\varphi : V \to W$ and $v \in V$. Therefore, we define $g(\varphi) = g_W \circ \varphi \circ g_V^{-1}$.

With these definitions, the isomorphism $\mathrm{Hom}(V, W) \cong V^* \otimes W$ becomes an isomorphism of representations. Also, the action on V^* can be viewed as a special case of $\mathrm{Hom}(V, W)$, namely the case where $W = k$ and G acts on W trivially.

1.3 complete reducibility and Schur's Lemma

By Complete Reducibility, we can consider irreps to be the building blocks of our theory:

Theorem 1.2 (Complete Reducibility)**.** Let G be a finite group. For any (complex) representation V with a subrepresentation W, there exists a subrepresentation U such that $V = W \oplus U$. Therefore, any finite-dimensional representation of G is a direct sum of irreducible representations.

Proof sketch. One can construct a Hermitian inner product $H : V \times V \to \mathbb{C}$ invariant under the action of G, meaning $H(gv, gw) = H(v, w)$ for all $g \in G$ and $v, w \in V$, for instance by taking any Hermitian inner product H_0 on V and "averaging" H_0 across the elements of G.

With respect to H, the operator $g : V \to V$ is unitary for all $g \in G$. Then[1] $g(W) \subseteq W$ implies $g(W^\perp) \subseteq W^\perp$, and the theorem follows by taking $U = W^\perp$.

Or, to avoid the use of a Hermitian inner product and to extend this result to any fields of characteristic zero, take any complement U_0 and "average" the projection $\pi_0 : V \to W$ with kernel U_0. The resulting projection is G-equivariant, and its kernel is invariant under G. □

Lemma 1.3 (Schur's Lemma)**.** Let G be a finite group, and let V and W be irreducible complex representations of G. If $\phi : V \to W$ is a homomorphism of representations, then either ϕ is an isomorphism or $\phi = 0$. Moreover, if V is finite-dimensional and $V = W$, then $\phi = \lambda I$ for some $\lambda \in \mathbb{C}$.

In other words, the dimension of $\mathrm{Hom}_G(V, W)$ is 0 (the space is trivial) or 1 (the space is spanned by an isomorphism and V is nontrivial).

Proof. The kernel of ϕ is a subrepresentation of V, and is therefore trivial or all of V. Meanwhile, im ϕ is a subrepresentation of W, and is therefore trivial or all of W. If $\ker \phi = V$ or im $\phi = 0$, then $\phi = 0$. Otherwise, $\ker \phi = 0$, meaning ϕ is injective, and im $\phi = W$, meaning ϕ is surjective.

We move on to the second statement. Since $\mathbb{C}$ is algebraically closed, the map $\phi : V \to V$ must have an eigenvalue λ. The λ-eigenspace U of ϕ is invariant under G, as $\phi(gv) = g\phi(v) = \lambda gv$ for any $g \in G$ and $v \in U$. Because U is nonzero and V is irreducible, in fact $U = V$ and $\phi = \lambda I$. □

We will see from character theory that G has finitely many non-isomorphic irreducible representations $V_1, \ldots, V_k$. By Complete Reducibility, any finite-dimensional representation V has a decomposition $V = \bigoplus V_i^{\oplus a_i}$. Moreover, by Schur's Lemma,

$$\dim \mathrm{Hom}_G(V, V_i) = \sum_{j=1}^{k} a_j \dim \mathrm{Hom}_G(V_j, V_i) = a_i \dim \mathrm{Hom}_G(V_i, V_i) = a_i,$$

proving that the decomposition of V into irreducibles is unique up to isomorphism.

1.4 abelian groups revisited

Let G be a finite abelian group, and let V be a finite-dimensional representation of G. We already encountered this setup in lecture, but equipped with Schur's Lemma, we have an alternative way of deducing the irreducible representations.

Each $g \in G$ defines a linear transformation $g : V \to V$. Because G is abelian, this map is G-equivariant—in other words, it is a homomorphism not just of vector spaces but of G-representations. Indeed, for all $h \in G$ and $v \in V$, we have $g(h(v)) = h(g(v))$. If V is irreducible, then it follows from Schur's Lemma that g acts on V by multiplication by

[1] Recall that unitary/orthogonal operators T preserve orthogonal complements of invariant subspaces. Suppose $W \subseteq V$ is invariant under T; take $u \in W^\perp$ and any $w \in W$. Unitary operators are invertible, and $w' = T^{-1}(w)$ belongs to U by assumption. Thus, $\langle Tu, w\rangle = \langle Tu, Tw'\rangle = \langle u, w'\rangle = 0$.

some scalar, a transformation that preserves subspaces. In particular, *any* subspace of V is invariant under G, so V must be one-dimensional to be irreducible.

We conclude that an irreducible representation of G is the same thing as a homomorphism $G \to \mathbb{C}^*$. In lecture, we used the classification of finitely generated abelian groups to specify such homomorphisms on generators.

2 characters

2.1 motivation

Our first examples suggest that the essential information of a representation $\rho : G \to GL(V)$ is captured in the eigenvalues of the operators $\rho(g) : V \to V$. How can we efficiently specify the eigenvalues of all of the elements of G? As it turns out, it is enough to specify the sum of the eigenvalues of each element of G.

There is a one-to-one correspondence between unordered n-tuples $(\lambda_1, \ldots, \lambda_n)$ of complex numbers with repetitions allowed and ordered n-tuples $(\mu_1, \ldots, \mu_n)$ of complex numbers, given in the forwards direction by taking the coefficients of $\prod_i (x - \lambda_i)$ and given in the backwards direction by taking the roots of the polynomial $x^n - \mu_1 x^{n-1} + \cdots + (-1)^n \mu_n$. In terms of the *elementary symmetric polynomials*

$$\sigma_k(x_1, \ldots, x_n) := \sum_{1 \le i_1 < \cdots < i_\alpha \le k} x_{i_1} \cdots x_{i_k},$$

the coefficient of x^{n-k} in $\prod_i (x - \lambda_i)$ is $(-1)^k \sigma_k(\lambda_1, \ldots, \lambda_n)$.

The symmetric group S_n acts on the polynomial ring $R := \mathbb{C}[x_1, \ldots, x_n]$ by permuting the x_i's. This action respects the structure of the ring: for any permutation $\sigma \in S_n$ and polynomials $f, g \in R$, we have $\sigma(fg) = \sigma(f)\sigma(g)$ and $\sigma(f + g) = \sigma(f) + \sigma(g)$. From this observation, it is not hard to see that the elements of R fixed by S_n form a subring of R, which we call *the ring of symmetric polynomials* $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$.

Theorem 2.1. Over $\mathbb{C}$, the elementary symmetric polynomials generate the ring of symmetric polynomials in n variables. We have an isomorphism

$$\mathbb{C}[x_1, \ldots, x_n]^{S_n} \cong \mathbb{C}[\sigma_1, \ldots, \sigma_n].$$

Another class of symmetric polynomials called the *power sums* $\tau_k(x_1, \ldots, x_n) := \sum_{i=1}^n x_i^k$ generate the ring of symmetric polynomials:

Theorem 2.2. Over $\mathbb{C}$, the power sums $\tau_1, \tau_2, \ldots, \tau_n$ generate the ring of symmetric polynomials in n variables. In particular, there is an isomorphism

$$\mathbb{C}[x_1, \ldots, x_n]^{S_n} \cong \mathbb{C}[\tau_1, \ldots, \tau_n].$$

Time to trace through these identifications. If the eigenvalues of g are $\{\lambda_1, \ldots, \lambda_n\}$, then the eigenvalues of g^k are $\{\lambda_1^k, \ldots, \lambda_n^k\}$. The powers of g appear as other elements of

the group, so specifying the sum of the eigenvalues of every $h \in G$ is enough to determine $\sum_{i=1}^{n} \lambda_i^k$ for each $k \in \{1, \ldots, n\}$. In terms of symmetric polynomials, we have the information $\tau_k(\lambda_1, \ldots, \lambda_n)$ for $1 \leq k \leq n$. By Theorems 4.1 and 4.2, this determines $\sigma_k(\lambda_1, \ldots, \lambda_n)$ for all $1 \leq k \leq n$, and by the correspondence we described, these elementary symmetric polynomials fully determine the λ_i.

2.2 definition and properties

Now that we're working with characters, we're going to assume for the remainder of the handout that G is a finite group and all representations are finite-dimensional and complex.

Definition 2.1. Let V be a representation of G. The *character* $\chi_V : G \to \mathbb{C}$ is the map $g \mapsto \mathrm{tr}(g : V \to V)$.

In general, χ_V has no reason to be a homomorphism. However, it takes the same value on conjugate elements of G because the trace of a linear operator is invariant under conjugation. For this reason, if we let $\mathcal{C}$ denote the set of conjugacy classes of G, then χ_V can be viewed as a map belonging to the space $\{\text{functions } \mathcal{C} \to \mathbb{C}\} = \mathbb{C}^{\mathcal{C}}$. We call χ_V a *class function*.

Proposition 2.3. For representations V, W of G, the character has the following properties:

(1) $\chi_V(e) = \dim(V)$

(2) $\chi_{V \oplus W} = \chi_V + \chi_W$

(3) $\chi_{V \otimes W} = \chi_V \cdot \chi_W$

(4) $\chi_{V^*} = \overline{\chi_V}$

(5) $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V(g)}\chi_W(g)$

(6) $\chi_{Sym^2(V)}(g) = \frac{\chi_V(g)^2 + \chi_V(g^2)}{2}$

(7) $\chi_{\wedge^2 V}(g) = \frac{\chi_V(g)^2 - \chi_V(g^2)}{2}$

Proof. (1): $\chi_V(e) = \mathrm{tr}(\mathrm{id}_V) = \dim V$.

(2)-(4): Fix $g \in G$. Fix an eigenbasis of $\{v_i\}$ of V for $g : V \to V$, and fix an eigenbasis $\{w_j\}$ of W for $g : W \to W$; let $\{\lambda_i\}$ and $\{\mu_j\}$ denote the corresponding eigenvalues. The direct sum $V \oplus W$ has an eigenbasis $\{v_i\} \cup \{w_j\}$ with eigenvalues preserved, and $V \otimes W$ has an eigenbasis $\{v_i \otimes w_j\}$ with eigenvalues $\{\lambda_i \mu_j\}$. The dual V^* has an eigenbasis $\{v_i^*\}$ with eigenvalues $\lambda_i^{-1} = \overline{\lambda_i}$ (remember the λ_i are unit complex numbers).

(5) is a consequence of (3), (4), and the isomorphism $\mathrm{Hom}(V, W) \cong V^* \otimes W$. Use suitable eigenbases for $\mathrm{Sym}^2(V)$ and $\wedge^2(V)(g)$ to work out (6) and (7). □

2.3 projection operator, Hermitian inner product, consequences

A representation V of G contains some number of copies of the one-dimensional trivial representation V_0. The subspace $V_0^{\oplus a_0} \subseteq V$ can be characterized as the subspace $V^G \subseteq V$ consisting of the vectors $v \in V$ fixed under G; that is, the vectors v such that that $gv = v$ for all $g \in G$. We showed that the operator

$$\varphi = \frac{1}{|G|} \sum_{g \in G} g : V \to V$$

is a homomorphism of representations and a projection onto V^G. As a consequence,

$$\dim V^G = tr(\varphi) = \frac{1}{|G|} \sum_{g \in G} tr(g) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g).$$

Now, for representations V, W of G, we can apply the above equality to the representation $\mathrm{Hom}(V, W)$. On your homework, you verified that $\mathrm{Hom}(V, W)^G$ is exactly the subspace $\mathrm{Hom}_G(V, W) \subseteq \mathrm{Hom}(V, W)$ of G-equivariant maps $V \to W$. We have

$$\dim \mathrm{Hom}_G(V, W) = \dim \mathrm{Hom}(V, W)^G = \frac{1}{|G|} \sum_{g \in G} \chi_{\mathrm{Hom}(V,W)}(g) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \chi_W(g).$$

To understand the important consequences of this result, we define an Hermitian inner product on the space $\mathbb{C}^{\mathcal{C}}$ of class functions on G by

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \cdot \beta(g).$$

In terms of this Hermitian inner product, we proved for representations V, W of G that

$$\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_G(V, W).$$

If V and W are irreducible, then Schur's Lemma implies

$$\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the characters of the irreducible representations are orthonormal with respect to our Hermitian inner product on $\mathbb{C}^{\mathcal{C}}$. Orthonormality implies linear independence, bounding the number of irreps of G by the number of conjugacy classes of G.

We will soon prove that

Theorem 2.4. The characters of the irreducible representations of G span the space of class functions $\mathbb{C}^{\mathcal{C}}$ on G.

The important implication is that the characters of the irreps form not just an orthonormal set but an *orthonormal basis* for the space of class functions on G. As a result,

Corollary 2.5. The number of irreducible representations of G is equal to the number of conjugacy classes of G.

Denote by $V_0, V_1, \ldots, V_k$ the distinct irreps of G where V_0 is the one-dimensional trivial representation. Any representation V has a decomposition $V = V_0^{\oplus a_0} \oplus \cdots \oplus V_k^{\oplus a_k}$.

Corollary 2.6. A representation is determined by its character. As seen earlier,

$$\langle \chi_V, \chi_{V_i} \rangle = \dim \operatorname{Hom}_G(V, V_i)$$

returns a_i, the number of copies of V_i in the decomposition of V into irreducibles. Notice that $\dim V^G = \dim V_0^{\oplus a_0} = \langle \chi_V, \chi_{V_0} \rangle$.

Corollary 2.7. Observe that

$$\dim \operatorname{Hom}_G(V, V) = \langle \chi_V, \chi_V \rangle = \sum_{i,j=0}^{k} a_i a_j \langle \chi_{V_i}, \chi_{V_j} \rangle = \sum_{i=0}^{k} a_i^2 \langle \chi_{V_i}, \chi_{V_i} \rangle = \sum a_i^2.$$

In particular, V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.

3 permutation representations

Permutation representations are those associated to an action $G \to \operatorname{Perm}(S)$ of a finite group G on a finite set S. One defines an $|S|$-dimensional complex vector space V: start with a basis $\{e_s\}_{s \in S}$, and set $V := \{\sum_{s \in S} c_s e_s \mid c_s \in \mathbb{C}\}$. In other words, we define V to be the vector space of functions $S \to \mathbb{C}$.

An element of $g \in G$ acts linearly on V via the mapping $e_s \mapsto e_{gs}$, and this action of G gives V the structure of a G-representation. In the given basis, each $g : V \to V$ is represented by a permutation matrix. The character $\chi_V(g)$ is the number of 1s along the diagonal of the associated permutation matrix, i.e. the number of the basis vectors e_s fixed by g, i.e. the number of fixed points $|S^g|$.

3.1 the regular representation

We've studied some special cases of this construction, one being the *regular representation* R associated to the action of G on itself by left multiplication (concretely, $g(e_h) = e_{gh}$ for elements $g, h \in G$). The regular representation of G is faithful, which implies that any every finite abstract group can be embedded in a linear group GL_n where $n = |G|$. (This is analogous to how the existence of this faithful group action $G \to \operatorname{Perm}(G)$ implies that every finite group can be realized as a subgroup of S_n.)

The character of the regular representation is

$$\chi_R(g) = \begin{cases} |G| & \text{if } g = e \\ 0 & \text{else.} \end{cases}.$$

Let $V_1, \ldots, V_k$ denote the irreps of G. From the calculation

$$\langle \chi_R, \chi_{V_i} \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_R(g)} \chi_{V_i}(g) = \frac{1}{|G|} \left(\overline{\chi_R(e)} \chi_{V_i}(e) \right) = \frac{1}{|G|} (|G| \cdot \dim V_i) = \dim V_i,$$

we see that the decomposition

$$R = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$$

of R as a direct sum of the V_i has $a_i = \dim V_i$. Notice that every irreducible representation of G appears as a subrepresentation of the regular representation.

We deduce from this decomposition that the dimensions of the irreducible representations of G are controlled by the size of the group. From direct calculation, we have

$$\langle \chi_R, \chi_R \rangle = \frac{1}{|G|} \left(\overline{\chi_R(e)} \chi_R(e) \right) = \frac{|G|^2}{|G|} = |G|,$$

while by Corollary 2.7,

$$\langle \chi_R, \chi_R \rangle = \sum a_i^2 = \sum (\dim\, V_i)^2.$$

Thus, if $V_1, \ldots, V_k$ are the irreducible representations of G, then

$$|G| = \sum_{i=1}^{k} (\dim\, V_i)^2.$$

3.2 the standard representation

The symmetric group S_n acts on $\mathbb{C}^n$ by permutations of the standard basis $\{e_i\}$. Denote this n-dimensional permutation representation by P. The subspace

$$U = \{(z_1, \ldots, z_n) \mid z_1 = \cdots = z_n\} = \text{span}(e_1 + \cdots + e_n)$$

is invariant under this action; P contains a copy of the one-dimensional trivial representation.

The vectors in the complement $V := U^\perp$ are the $w = \sum a_i e_i$ with $\sum a_i = 0$. For any $\sigma \in S_n$ and $v = \sum a_i e_i \in V$, we have $\sum a_{\sigma(i)} = \sum a_i = 0$. The space V is invariant under S_n, and is therefore a representation of S_n in its own right. We call V the *standard representation* of S_n. Notice that V is spanned by the $n-1$ vectors $u_i = \{e_i - e_n\}_{1 \le i \le n-1}$, as any element w of V can be written as

$$w = a_1 e_1 + \cdots + a_{n-1} e_n - (a_1 + \cdots + a_{n-1}) e_n = \sum_{i=1}^{n-1} a_i u_i.$$

In fact, the u_i are linearly independent and thus form a basis for V.

Proposition 3.1. The standard representation is irreducible.

Proof. Let $v = \sum c_i e_i$ be any nonzero vector in V. Define $W \subseteq V$ to be the subrepresentation

of V generated by v, meaning the smallest subrepresentation of V containing v. (Concretely, W is the span of the vectors $\{gv\}_{g\in G}$.) We show that $W = V$.

There must exist k, ℓ such that $c_k \neq c_\ell$. Letting $\sigma_{k\ell}$ denote the transposition in S_n swapping k and ℓ, we have

$$\frac{1}{c_k - c_\ell}(v - \sigma_{k\ell}(v)) = \frac{1}{c_k - c_\ell}(c_k - c_\ell)(e_k - e_\ell) = e_k - e_\ell.$$

For each $i \in \{1, \ldots, n-1\}$, the vector $\sigma_{(ik)}\sigma_{(\ell n)}(e_k - e_\ell) = u_i$ belongs to W. We are done because the u_i span V. □

Using the decomposition $P = U \oplus V$, we can express the character of V in terms of fixed points where $S = \{1, 2, \ldots, n\}$:

$$\chi_V(g) = \chi_P(g) - \chi_U(g) = |S^g| - 1.$$

4 Miscellaneous observations

- The first column of a character table lists $\chi_V(e) = \dim V$. Recall that $|G| = \sum(\dim V_i)^2 = \sum \chi_{V_i}(e)^2$.
- If V is a one-dimensional (hence irreducible) representation of G and $g \in G$ has order n, then $\chi_V(g)$ is an n-th root of unity.
- If there is only one irreducible representation W of dimension n, then $W^* \cong W$, which implies that $\chi_W = \overline{\chi_W}$ for all $g \in G$. Moreover, for any one-dimensional representation U, we have $W \otimes U \cong W$, and thus $\chi_W(g)\chi_U(g) = \chi_W(g)$ for all $g \in G$. The consequence we've been using is that $\chi_U(g) \neq 1$ implies $\chi_W(g) = 0$.
- If the character of a representation V is real-valued ($\chi_V(g) \in \mathbb{R}$ for all $g \in G$), then $V \cong V^*$. Hence, if χ_{V_i} is real-valued for every irreducible representation V_i, then every representation of G is isomorphic to its dual.

I won't write this up because tables are easier to draw on a board than to LaTeX, but an exercise I recommend is to take the standard representation V of S_4 find the decomposition of $V \otimes V$ into irreducibles. Answer in the footnotes.[2]

[2] In lecture notation, $V \otimes V$ is isomorphic to $U \oplus V \oplus V' \oplus W$.