rep theory recap

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1 basics

Definition 1.1. Let G be a group. A representation of G is a vector space V along with a homomorphism $\rho : G \to GL(V)$.

In 55, we are concerned almost exclusively with finite groups and finite-dimensional vector spaces over $\mathbb{C}$. Complete reducibility, one of our most important statements, depends upon working over a characteristic zero field. We use algebraic closure to prove Schur's Lemma and the fact that every operator $g : V \to V$ is diagonalizable. Our approach can be somewhat modified to compensate for the loss of algebraic closure, but it completely breaks down without the characteristic zero assumption. Representation theory in characteristic p looks totally different. Another technicality: some my statements will only be accurate modulo the case $V = 0$.

If ρ sends all elements of G to the identity transformation in $GL(V)$, then it is called the *trivial representation* in dimension $n = \dim V$. At the opposite end, injective homomorphisms $G \to GL(V)$ are embeddings (*faithful* representations); with no loss of information, we can view G as a linear group. I was told that before mathematicians came up with the notion an abstract group, the groups they thought about were just sets of invertible matrices closed under inverses and matrix multiplication. The study of representation theory is more general—we want to describe all of the ways that a given abstract group G can be mapped to a linear group GL_n.

Definition 1.2. Let V and W be representations of G. A linear map $\phi : V \to W$ is a *homomorphism of representations* or a *G-equivariant linear map* if it respects the action of G, meaning that for all $g \in G$ and $v \in V$, we have $\phi(g_V \cdot v) = g_W \cdot \phi(v)$.

The set of G-equivariant linear maps $V \to W$, denoted $\mathrm{Hom}_G(V, W)$, forms a vector subspace of $\mathrm{Hom}(V, W)$.

Definition 1.3. If V is a representation of G, then a *subrepresentation* of V is a subspace $W \subseteq V$ invariant under the action of G, meaning $g(W) = W$ for all $g \in G$.

A subrepresentation is a representation of G in its own right, one that happens to be a subspace of V. We could have defined a subrepresentation of V equivalently as a representation W of G along with a G-equivariant injection $W \hookrightarrow V$ (e.g. the inclusion, if W is already realized as a subspace of V).

Irreducible representations are those that cannot be decomposed any further:

Definition 1.4. A representation V of G is *irreducible* if it has no subrepresentations except for $W = 0$ and $W = V$.

Because any subspace of a trivial representation is invariant under G, the only irreducible trivial representation is the trivial representation of dimension 1.

Lemma 1.5. If G is a finite group and V is finite-dimensional, complex representation of G, then each $\rho(g) : V \to V$ is diagonalizable, and the eigenvalues of g are roots of unity.

Proof. Observe that the operator $\rho(g) \in GL(V)$ is of finite order (its order divides the order of g). From here, use the minimal polynomial as we did in lecture. A different proof can be found in Artin under Theorem 4.7.14. □

2 constructions

Given representations V and W of a group G, we can turn $V \oplus W$ into a G-representation by defining $g(v, w) := (gv, gw)$. The actions on the V and W factors don't interact. Similarly, $V \otimes W$ can be given the structure of a G-representation by setting $g(v \otimes w) := gv \otimes gw$.

To define the dual of a representation $\rho : G \to GL(V)$, consider the natural bilinear pairing $V \times V^* \to \mathbb{C}$ given by $(v, \ell) \mapsto \ell(v)$. We want the action of G on V^* to respect this pairing, meaning $(g_V(v), g_{V^*}(\ell)) \mapsto \ell(v)$. So, we define $\rho^*(g) = {}^t\rho(g^{-1}) : V^* \to V^*$, which satisfies

$$g_{V^*}(\ell)(g_V(v)) = (\ell \circ g_V^{-1})(g_V(v)) = \ell(g_V^{-1} g_V(v)) = \ell(v).$$

Similarly, when we give $\mathrm{Hom}(V, W)$ the structure of a G-representation, we want the action to satisfy $g(\varphi)(gv) = \varphi(v)$ for any $\varphi : V \to W$ and $v \in V$. Therefore, we define $g(\varphi) = g_W \circ \varphi \circ g_V^{-1}$.

With these definitions, the isomorphism $\mathrm{Hom}(V, W) \cong V^* \otimes W$ becomes an isomorphism of representations. Also, the action on V^* can be viewed as a special case of $\mathrm{Hom}(V, W)$, namely the case where $W = k$ and G acts on W trivially.

3 complete reducibility and Schur's Lemma

By Complete Reducibility, we can consider irreps to be the building blocks of our theory:

Theorem 3.1 (Complete Reducibility)**.** Let G be a finite group. For any (complex) representation V with a subrepresentation W, there exists a subrepresentation U such that $V = W \oplus U$. Therefore, any finite-dimensional representation of G is a direct sum of irreducible representations.

Proof sketch. One can construct a Hermitian inner product $H : V \times V \to \mathbb{C}$ invariant under the action of G, meaning $H(gv, gw) = H(v, w)$ for all $g \in G$ and $v, w \in V$, for instance by taking any Hermitian inner product H_0 on V and "averaging" H_0 across the elements of G.

With respect to H, the operator $g : V \to V$ is unitary for all $g \in G$. Then[1] $g(W) \subseteq W$ implies $g(W^\perp) \subseteq W^\perp$, and the theorem follows by taking $U = W^\perp$.

Or, to avoid the use of a Hermitian inner product and to extend this result to any fields of characteristic zero, take any complement U_0 and "average" the projection $\pi_0 : V \to W$ with kernel U_0. The resulting projection is G-equivariant, and its kernel is invariant under G. □

Lemma 3.2 (Schur's Lemma)**.** Let G be a finite group, and let V and W be irreducible complex representations of G. If $\phi : V \to W$ is a homomorphism of representations, then either ϕ is an isomorphism or $\phi = 0$. Moreover, if V is finite-dimensional and $V = W$, then $\phi = \lambda I$ for some $\lambda \in \mathbb{C}$.

In other words, the dimension of $\mathrm{Hom}_G(V, W)$ is 0 (the space is trivial) or 1 (the space is spanned by an isomorphism and V is nontrivial).

Proof. The kernel of ϕ is a subrepresentation of V, and is therefore trivial or all of V. Meanwhile, im ϕ is a subrepresentation of W, and is therefore trivial or all of W. If $\ker \phi = V$ or im $\phi = 0$, then $\phi = 0$. Otherwise, $\ker \phi = 0$, meaning ϕ is injective, and im $\phi = W$, meaning ϕ is surjective.

We move on to the second statement. Since $\mathbb{C}$ is algebraically closed, the map $\phi : V \to V$ must have an eigenvalue λ. The λ-eigenspace U of ϕ is invariant under G, as $\phi(gv) = g\phi(v) = \lambda gv$ for any $g \in G$ and $v \in U$. Because U is nonzero and V is irreducible, in fact $U = V$ and $\phi = \lambda I$. □

We will see from character theory that G has finitely many non-isomorphic irreducible representations $V_1, \ldots, V_k$. By Complete Reducibility, any finite-dimensional representation V has a decomposition $V = \bigoplus V_i^{\oplus a_i}$. Moreover, by Schur's Lemma,

$$\dim \mathrm{Hom}_G(V, V_i) = \sum_{j=1}^{k} a_j \dim \mathrm{Hom}_G(V_j, V_i) = a_i \dim \mathrm{Hom}_G(V_i, V_i) = a_i,$$

proving that the decomposition of V into irreducibles is unique up to isomorphism.

3.1 abelian groups revisited

Let G be a finite abelian group, and let V be a finite-dimensional representation of G. We already encountered this setup in lecture, but equipped with Schur's Lemma, we have an alternative way of deducing the irreducible representations.

Each $g \in G$ defines a linear transformation $g : V \to V$. Because G is abelian, this map is G-equivariant—in other words, it is a homomorphism not just of vector spaces but of G-representations. Indeed, for all $h \in G$ and $v \in V$, we have $g(h(v)) = h(g(v))$. If V is irreducible, then it follows from Schur's Lemma that g acts on V by multiplication by

[1] Recall that unitary/orthogonal operators T preserve orthogonal complements of invariant subspaces. Suppose $W \subseteq V$ is invariant under T; take $u \in W^\perp$ and any $w \in W$. Unitary operators are invertible, and $w' = T^{-1}(w)$ belongs to U by assumption. Thus, $\langle Tu, w\rangle = \langle Tu, Tw'\rangle = \langle u, w'\rangle = 0$.

some scalar, a transformation that preserves subspaces. In particular, *any* subspace of V is invariant under G, so V must be one-dimensional to be irreducible.

We conclude that an irreducible representation of G is the same thing as a homomorphism $G \to \mathbb{C}^*$. In lecture, we used the classification of finitely generated abelian groups to specify such homomorphisms on generators.

4 motivation for characters

Our first examples suggest that the essential information of a representation $\rho : G \to GL(V)$ is captured in the eigenvalues of the operators $\rho(g) : V \to V$. How can we efficiently specify the eigenvalues of all of the elements of G? As it turns out, it is enough to specify the sum of the eigenvalues of each element of G.

There is a one-to-one correspondence between unordered n-tuples $(\lambda_1, \ldots, \lambda_n)$ of complex numbers with repetitions allowed and ordered n-tuples $(\mu_1, \ldots, \mu_n)$ of complex numbers, given in the forwards direction by taking the coefficients of $\prod_i (x - \lambda_i)$ and given in the backwards direction by taking the roots of the polynomial $x^n - \mu_1 x^{n-1} + \cdots + (-1)^n \mu_n$. In terms of the *elementary symmetric polynomials*

$$\sigma_k(x_1, \ldots, x_n) := \sum_{1 \le i_1 < \cdots < i_\alpha \le k} x_{i_1} \cdots x_{i_k},$$

the coefficient of x^{n-k} in $\prod_i (x - \lambda_i)$ is $(-1)^k \sigma_k(\lambda_1, \ldots, \lambda_n)$.

The symmetric group S_n acts on the polynomial ring $R := \mathbb{C}[x_1, \ldots, x_n]$ by permuting the x_i's. This action respects the structure of the ring: for any permutation $\sigma \in S_n$ and polynomials $f, g \in R$, we have $\sigma(fg) = \sigma(f)\sigma(g)$ and $\sigma(f + g) = \sigma(f) + \sigma(g)$. From this observation, it is not hard to see that the elements of R fixed by S_n form a subring of R, which we call *the ring of symmetric polynomials* $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$.

Theorem 4.1. Over $\mathbb{C}$, the elementary symmetric polynomials generate the ring of symmetric polynomials in n variables. We have an isomorphism

$$\mathbb{C}[x_1, \ldots, x_n]^{S_n} \cong \mathbb{C}[\sigma_1, \ldots, \sigma_n].$$

Another class of symmetric polynomials called the *power sums* $\tau_k(x_1, \ldots, x_n) := \sum_{i=1}^n x_i^k$ generate the ring of symmetric polynomials:

Theorem 4.2. Over $\mathbb{C}$, the power sums $\tau_1, \tau_2, \ldots, \tau_n$ generate the ring of symmetric polynomials in n variables. In particular, there is an isomorphism

$$\mathbb{C}[x_1, \ldots, x_n]^{S_n} \cong \mathbb{C}[\tau_1, \ldots, \tau_n].$$

Time to trace through these identifications. If the eigenvalues of g are $\{\lambda_1, \ldots, \lambda_n\}$, then the eigenvalues of g^k are $\{\lambda_1^k, \ldots, \lambda_n^k\}$. The powers of g appear as other elements of the group, so specifying the sum of the eigenvalues of every $h \in G$ is enough to determine $\sum_{i=1}^n \lambda_i^k$ for each $k \in \{1, \ldots, n\}$. In terms of symmetric polynomials, we have the information

$\tau_k(\lambda_1, \ldots, \lambda_n)$ for $1 \leq k \leq n$. By Theorems 4.1 and 4.2, this determines $\sigma_k(\lambda_1, \ldots, \lambda_n)$ for all $1 \leq k \leq n$, and by the correspondence we described, these elementary symmetric polynomials fully determine the λ_i.