Week 6 Review Session Notes, Part 2

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1 Orthogonal and self-adjoint operators

Orthogonal operators on an inner product space are the linear transformations that preserve lengths and angles.

Definition 1.1. An operator $T : V \to V$ on a real inner product space is called *orthogonal* if it respects the inner product, meaning that for any $v, w \in V$,

$$\langle Tv, Tw\rangle = \langle v, w\rangle.$$

Orthogonal operators are invertible (they map orthonormal bases to orthonormal bases). The inverse of an orthogonal operator is orthogonal, as is the composition of two orthogonal operators. Whereas self-adjoint operators form a vector subspace of $\mathrm{End}(V)$, orthogonal operators form a subgroup of $GL(V)$, denoted $O(V)$. Self-adjoint operators are not a group under composition, while orthogonal operators are not generally closed under addition and certainly not under scaling.

The following proposition is essential in the proof of the Spectral Theorem, and is generally useful for decompositions. The same statement holds over a Hermitian inner product space.

Proposition 1.1 (Axler 7.28)**.** Suppose that $T : V \to V$ is a self-adjoint operator on an inner product space V. If $U \subseteq V$ is a subspace invariant under T, then:

(1) $U^\perp$ is invariant under T.

(2) The restriction $T|_U$ is self-adjoint.

(3) The restriction $T|_{U^\perp}$ is self-adjoint.

Proof. (1) Let v be a vector in $U^\perp$. For each $u \in U$ we have $\langle Tv, u\rangle = \langle v, Tu\rangle$ by self-adjointness and $\langle v, Tu\rangle = 0$ because Tu belongs to U. Hence, $Tv \in U^\perp$, and this holds for any $v \in U^\perp$, so $T(U^\perp) \subset U^\perp$.

(2) The restriction of an inner product to an invariant subspace is still an inner product, and $\langle (T|_U)u, v\rangle = \langle Tu, v\rangle = \langle u, Tv\rangle = \langle u, (T|_U)v\rangle$. We need invariance so that $Tu, Tv \in U$.

(3) The result of (1) allows us to replace U by $U^\perp$ in (2).

□

Lemma 1.2. Suppose $T : V \to V$ is a self-adjoint operator on a real finite-dimensional inner product space $(V, \langle \cdot, \cdot \rangle)$. Then T has an eigenvalue.

Proof. The operators $1, T, T^2, \ldots$ cannot be linearly independent, so T satisfies a polynomial relation with coefficients in $\mathbb{R}$, i.e. there exists $P(x) \in \mathbb{R}[x]$ such that $P(T) = 0$. Using the Fundamental Theorem of Algebra, we decompose the polynomial into linear and quadratic factors: we can write $P(x) = c\prod(x-\lambda_i)\prod Q_j(x)$ where each quadratic $Q_j(x) = x^2+b_jx+c_j$ has no real roots. By lecture theory, i.e. Proof by Prof. Auroux, the linear operators $Q_j(T)$ are invertible. From $P(T) = c\prod(T - \lambda_i)\prod Q_j(T) = 0$, we see that some $T - \lambda_i$ has a kernel. □

Theorem 1.3 (Real Spectral Theorem, Axler 7.29)**.** Suppose $T : V \to V$ is a self-adjoint operator on a finite-dimensional real inner product space $(V, \langle \cdot, \cdot \rangle)$. Then V has an orthonormal basis of eigenvectors of T (in fact, this is an equivalence). In other words, T is diagonalizable in an orthonormal basis.

Proof. By Lemma 1.2, T has an eigenvector vector v_1. We scale so that $|v_1| = 1$. The subspace $U := \text{span}(v_1) \subseteq V$ is invariant under T; we have a direct sum decomposition $V = U \oplus U^\perp$. By Lemma 1.1, $U^\perp$ is invariant under T, and the restriction $T|_{U^\perp}$ is self-adjoint. We assume by induction that $U^\perp$ has an orthonormal basis of eigenvectors $v_2, \ldots, v_n$ of $T|_{U^\perp}$. Then $v_1, v_2, \ldots, v_n$ is an orthonormal basis of V comprised of eigenvectors of T. □

Independent of the spectral theorem, one can directly show that

Lemma 1.4. If $T : V \to V$ is a self-adjoint operator on a real inner product space V, then eigenvectors with different eigenvalues are orthogonal.

Proof. Suppose $v, w \in V$ are eigenvectors of T with eigenvalues λ, μ, respectively. Then $\lambda\langle v, w\rangle = \langle Tv, w\rangle = \langle v, Tw\rangle = \mu\langle v, w\rangle$, so $\lambda \neq \mu$ implies $\langle v, w\rangle = 0$. □

Geometrically, for any self-adjoint operator T on an n-dimensional inner product space V, we have a set of n orthogonal axes such that T acts along each axis by scaling.

Theorem 1.5 (Orthogonal operators)**.** Let $T : V \to V$ be an orthogonal operator on a finite-dimensional inner product space V. Then V admits a direct sum decomposition $V = \bigoplus U_i \oplus \bigoplus W_j$ where:

- each U_i and each W_j is invariant under T;
- the subspaces U_i and W_j are orthogonal to one another;
- each U_i is 1-dimensional and each W_j is two-dimensional;
- $T|_{U_i} = \pm I$ for each i, and $T|_{W_j}$ is a rotation for each j.

Sketch of proof. We proved back in Lecture 12 that V has either a one-dimensional or a two-dimensional subspace U that is invariant under T. On a 1-dimensional vector space, an orthogonal operator must be scaling by ± 1. On a two-dimensional vector space, an orthogonal operator is either a rotation or a reflection. If it is a reflection, then the vector space

decomposes into two one-dimensional ± 1 eigenspaces. From $T(U) \subset U$ and the property that T is an isomorphism, one shows (similar to the proof of 1.1) that $T(U^\perp) = U^\perp$ and $T|_{U^\perp}$ is orthogonal. Apply the inductive hypothesis to $T|_{U^\perp}$. $\square$

2 Hermitian inner products

A positive-definite bilinear form doesn't exist over a complex vector space V. Given a bilinear form $B : V \times V \to \mathbb{R}$ and any vector $v \in V$, we have $B(iv, iv) = -B(v, v)$; only one of $B(v, v)$ and $B(iv, iv)$ can be positive. Moreover, if $\dim V \geq 2$, there exists a nonzero vector $u \in V$ such that $B(u, u) = 0$. To fix the problem, we need to give up a quarter of bilinearity:

Definition 2.1. Let V be a complex vector space. A *Hermitian form* on V is a map $h : V \times V \to \mathbb{C}$ such that:

- h is linear in the first factor, meaning $h(u + v, w) = h(u, w) + h(v, w)$ and $h(\lambda u, v) = \lambda h(u, v)$;
- h is conjugate-linear in the second factor, meaning $h(u, v + w) = h(u, v) + h(u, w)$ and $h(u, \lambda v) = \overline{\lambda} h(u, v)$;
- h is conjugate symmetric, meaning $h(u, v) = \overline{h(v, u)}$.

If in addition h is positive-definite, meaning $h(v, v) \geq 0$ for all $v \in V$ and $h(v, v) = 0 \iff v = 0$, then we call h a *Hermitian inner product*.

Note: for all $v \in V$, conjugate-symmetry implies $h(v, v) = \overline{h(v, v)}$, so $h(v, v)$ is real for Hermitian forms, not just for inner products.

The "sesquilinearity" of h. A Hermitian inner product is linear with respect to multiplication by real and imaginary scalars in the first variable, and it is linear with respect to multiplication by real numbers in the second. However, we had to give up linearity with respect to multiplication by imaginary numbers in the second variable.

We can still associate to any $v \in V$ the linear form $h(\cdot, v) \in V^*$. This map $V \to V^*$ remains a bijection, but it is now complex-antilinear instead of linear, as $h(\cdot, \lambda v) = \overline{\lambda} h(\cdot, v)$.

On $\mathbb{C}^n$, there exists the standard Hermitian inner product, given by $h(u, v) = \sum_{i=1}^n z_i \overline{w_i}$ where $u = (z_1, \ldots, z_n), v = (w_1, \ldots, w_n)$. As in the case of real inner products, any n-dimensional Hermitian inner product space V has an orthonormal basis and is therefore isomorphic to $\mathbb{C}^n$ with the standard Hermitian inner product.

For operators on a complex inner product space, we use the same definition of self-adjoint, and we replace "orthogonal" by "unitary" (the Complex Spectral Theorem explains why we use the term unitary instead). Unitary operators likewise map orthonormal bases to orthonormal bases, and are therefore invertible.

Definition 2.2. Let $T : V \to V$ be a linear operator on a Hermitian inner product space V.

- T is *unitary* if it respects the inner product, i.e. $\langle Tv, Tw \rangle = \langle v, w \rangle$ for all $v, w \in V$. In other words, $T^* = T^{-1}$, as $\langle Tv, Tw \rangle = \langle v, w \rangle \iff \langle Tv, w \rangle = \langle v, T^{-1} w \rangle$.

- T is *self-adjoint* if $\langle Tv, w\rangle = \langle v, Tw\rangle$ for all $v, w \in V$. In other words, $T^* = T$.

Theorem 2.1 (Complex Spectral Theorem)**.** Let $T : V \to V$ be a linear operator on a finite-dimensional Hermitian inner product space. Then:

- If T is self-adjoint, then there exists an orthonormal basis for V consisting of eigenvectors of T, and all eigenvalues of T are real.
- If T is unitary, then there exists an orthonormal basis of V consisting of eigenvectors of T. All eigenvalues of T are complex numbers with absolute value 1.

Sketch of proof. For self-adjoint operators, we have $\lambda\langle v, v\rangle = \langle Tv, v\rangle = \langle v, Tv\rangle = \overline{\lambda}\langle v, v\rangle$, so $\lambda \in \mathbb{R}$. As for unitary operators, $\lambda\overline{\lambda}\langle v, v\rangle = \langle Tv, Tv\rangle = \langle v, v\rangle$, implying that $\overline{\lambda} = \lambda^{-1}$, which holds exactly when λ is a unit complex number.

For eigenvectors v, w with distinct eigenvalues μ, λ, respectively, it still holds when T is self-adjoint that $\lambda\langle v, w\rangle = \langle Tv, w\rangle = \langle v, Tw\rangle = \overline{\mu}\langle v, w\rangle = \mu\langle v, w\rangle \implies \langle v, w\rangle = 0$, just as in the real case. When T is unitary, w is an eigenvector of T^{-1} with eigenvalue μ^{-1}. We still have

$$\lambda\langle v, w\rangle = \langle Tv, w\rangle = \langle v, T^{-1}w\rangle = \overline{\mu^{-1}}\langle v, w\rangle = \mu\langle v, w\rangle,$$

and hence $\langle v, w\rangle = 0$.

The remainder of the proof is analogous to that of the Real Spectral Theorem; over $\mathbb{C}$, we already know that *any* linear operator has an eigenvalue. □

Our description of unitary operators over complex inner product spaces is much simpler than our description of orthogonal operators over real inner product spaces, which I interpret as one of many pieces of evidence that $\mathbb{C}$ is superior to $\mathbb{R}$.

3 Matrices :(

Let $T : V \to V$ be an operator on a real inner product space. In an orthonormal basis of V, the matrix representing T^* is the transpose of the matrix representing T. Therefore, T is self-adjoint if and only if the matrix of T is symmetric in an orthonormal basis of V. We can be more specific, if we want:

Lemma 3.1. T is self-adjoint if and only if T is represented by a diagonal matrix with respect to some orthonormal basis of V.

Proof. $\implies$: Take the basis given by the spectral theorem.

$\impliedby$: A diagonal matrix is equal to its transpose, so $T = T^*$. □

Lemma 3.2 (Orthogonal matrices)**.** If $T : V \to V$ is an orthogonal operator on a real inner product space V, then the matrix M representing T in an orthonormal basis of V satisfies $M^T M = I$.

Consider a linear operator $T : V \to V$ on a Hermitian inner product space. In an orthonormal basis, T^* is represented by the conjugate-transpose of the matrix representing T. Therefore, a self-adjoint operator is represented by a Hermitian matrix ($a_{ij} = \overline{a_{ji}}$).