Semidirect Products!

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1 Motivation

One of you a while ago asked me upon learning what cosets are if we could use them to build a sort of "coordinate system" for a group, breaking it down into components based on what coset of the subgroup you're in and what element of the subgroup you're multiplying by. I did not answer that question very well at the time I fear, but it turns out that the answer to that question is exactly what will lead us to the notion of semidirect product. So consider this my pennance.

The motivation here, then, is exactly that idea: Given a subgroup $N \leq G$ of some finite group G, can we somehow represent elements of G as pairs (n, h) for some index set H? Trivially, yes: just take the quotient set G/N and pick a set H of coset representatives, and then we indeed have a way of writing every element of G as a pair, namely writing it as nh for $n \in N$ and $h \in H$ for our index set H. As is, though, this is not particularly useful: almost everything involved is just sets and as such this kind of decomposition isn't going to be particularly enlightening, as it doesn't interact nicely with any of the structures we've been building over the course of the class. Moreover, our choice of index set was pretty arbitrary and unnatural, meaning this representation is by no means unique. So we're going to impose some restrictions on this construction to try and make it more tractable:

1. $N \trianglelefteq G$, that is N is a normal subgroup so that G/N actually has a group structure, and so that the decomposition into cosets doesn't rely on the rather arbitrary choice of left or right coset
2. $H \leq G$ is a subgroup
3. every coset of N contains a *unique* element of H, so that the representation is in fact unique

This set of conditions is basically so that our decomposition actually plays nicely with all the group structures we care about, allowing us to break things into component parts that respect our existing structure. Think of it as vaguely analogous to trying to capture how nice the idea of complementary subspaces in linear algebra was – we wanted to find a way of breaking a vector space V into U, U' so that we could represent members of V as pairs $(u, u') \in U \times U'$. There are *a priori* many ways to do this, but the one we wanted had a uniqueness condition $U \cap U' = \{0\}$ and the condition that U' was itself a vector space, so that we could actually apply the tool of linear algebra to the problem. To illustrate how much more natural these criteria make this construction, let's see how we can phrase them purely in terms of diagrams. First, a claim:

Proposition.

Under the conditions outlined above, $H \cong G/N$

Proof: We can give the isomorphism explicitly as follows:

$$\begin{aligned} \varphi : H &\xrightarrow{\sim} G/N \\ h &\mapsto hN = Nh \end{aligned}$$

This is a homomorphism because it is the restriction to a subgroup of the quotient map which is a homomorphism, injective because it maps h to its coset and by hypothesis each coset contains a *unique* element of H, and surjective because every coset contains *an* element of H. Note that this map being an isomorphism is in fact equivalent to our statement □

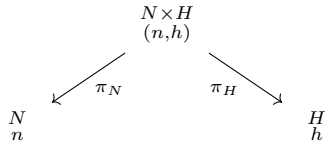
A fancy way of phrasing this is that we have an *exact sequence* of the form:

$$1 \to N \xhookrightarrow{\iota} G \xrightarrow{p} H \to 1$$

For a bit of review, that this sequence is *exact* means that the kernel of p and the image of ι coincide and similarly at each of the other points, which in practice means for a short exact sequence (one with only 3 terms) that ι is an injection and p a surjection. By first isomorphism theorem that there exists such an exact sequence is the same statement as N being normal[1] and $H \cong G/N$. Moreover, that H is isomorphic to G/N is exactly the statement that every coset of N contains a unique element of N. So indeed with these conditions we're able to describe our setup very naturally, by which I mean we're able to do it in terms of the groups themselves and homomorphisms between them rather than having to dig too deep into the elementwise structure. Returning now to our program however, you might hope that we can say something along the lines of saying that in this situation we have an isomorphism:

$$G \cong N \times H$$

After all, we most definitely have a bijection of sets $N \times H \to G$ by construction, since every element of G can be uniquely written as a product nh, so we can just send $(n, h) \mapsto nh$. This, however, cannot work in general, and here's why. Recall that we have for any product a pair of natural projection maps:



The kernel of π_N is exactly H, and of π_H exactly N, meaning that both N and H must be normal subgroups of G. But this is not in general true – in particular H has no general reason to be normal – and hence it cannot be true in general that G decomposes in this way. So although G is indeed in bijection with the set $N \times H$, the standard direct product cannot in general be the correct construction. Semidirect products are a way of endowing the set $N \times H$ with a group structure that makes this bijection into an isomorphism. So let's get into how this actually works.

[1] or rather in general there being a normal subgroup isomorphic to N

2 Semidirect Products

We seek to recover some kind of group law on the *set* $N \times H$ with respect to which the map defined above is an isomorphism. To do this, let's work some more in G, and then see if we can recover some kind of group structure on our representation of elements of g as products nh. That is, given $g_1 = n_1h_1, g_2 = n_2h_2$, how can we write g_1g_2 as a product of an element of N and an element of H? We can simply do this by some clever inserting of an $h_1h_1^{-1}$ and careful noting of what normality gives us:

$$\begin{aligned} g_1g_2 &= n_1h_1n_2h_2 \\ &= n_1h_1n_2eh_2 \\ &= n_1h_1n_2(h_1^{-1}h_1)h_2 \\ &= \underbrace{\underbrace{(n_1)}_{\in N}\ \underbrace{(h_1nh_1^{-1})}_{\in N \text{ by } N \text{ normal}}}_{\in N \text{ by } N \text{ a subgroup}}\ \underbrace{(h_1h_2)}_{\in H} \end{aligned}$$

So we have that if we enforce that the map $\varphi : (n, h) \mapsto nh$ is a homomorphism for some group structure on $N \times H$, the group law is *almost* $(n_1, h_1)(n_2, h_2) = (n_1n_2, h_1h_2)$, except that it differs by conjugation, in that the group law is the original group law in the second coordinate but composition of n_1 with *the conjugate of* n_2 *by* h_1 in the first, that is:

$$(n_1, h_1)(n_2, h_2) = (n_1 \cdot h_1n_2h_1^{-1}, h_1h_2)$$

So just define this as our group law, check that that's actually a group, and be done with it right? Not so fast. We're now trying to define a group law on the abstract set $N \times H$, without reference to the structure of G itself, since if we had to reference the existing structure of G this wouldn't be a very good general decomposition approach. So we need to find a way to phrase what's happening here in a way that doesn't make direct reference to the notion of multiplying elements of N with elements of H. In service of this, let's make an observation:

Proposition.

Given a finite group G, a normal subgroup $N \trianglelefteq G$, and a subgroup $H \leq G$, we have an action of $H \curvearrowright N$ by conjugation.[a]

Proof: That this is an action on N follows from the general fact that subgroups act by conjugation on the group alongside the fact that N is closed under conjugation meaning this action sends N to itself. Put another way, H acts on G via the restriction of the action of G on itself by conjugation to H and N is an orbit of this action, and we know in general that groups act individually on the orbits of actions. Moreover, notice that this action is not just by general permutations $\mathrm{Perm}(N)$ but rather by automorphisms; that is, the homomorphism $\rho : H \to \mathrm{Perm}(N)$ actually has image contained in the subgroup $\mathrm{Aut}(N)$ given by bijections $N \xrightarrow{\sim} N$ that are also group isomorphisms, and we have in fact a homomorphism:

$$\rho : H \to \mathrm{Aut}(N)$$

This kind of action, where it's a homomorphism not just to a group of bijections but to a group of structure-preserving bijections, is *very* important for the rest of the course.

[a] If you haven't seen it before, the notation $H \curvearrowright N$ can be read as "H acting on N." If you like this notation, the latex is \curvearrowright

So what we're really saying when we say that "the group law on $N \times H$ that makes φ an isomorphism is $(n_1, h_1)(n_2, h_2) = (n_1 \cdot h_1n_2h_1^{-1}, h_1h_2)$" is that for a particular action by automorphisms $\rho : H \to \mathrm{Aut}(N)$,

which we secretly know is given by $\rho(h)(n) = hnh^{-1}$ the conjugation action, the group law is given by:

$$(n_1, h_1)(n_2, h_2) = (n_1\rho(h_1)(n_2), h_1h_2)$$

But notice now that we can phrase the idea of an action by automorphisms independently of any preexisting group structure on $N \times H$, just by saying it's some homomorphism $\rho : H \to \mathrm{Aut}(N)$. Then every instance of multiplication in this expression can be defined in abstract without reference to the forbidden multiplication of N-elements with H-elements. Hence we can define a group law $\bullet$ this way in the abstract, and then what we've proven is that our situation with N a normal subgroup and H a subgroup with a unique representative in each coset is that we have an isomorphism with the group $(N \times H, \bullet)$. Let's make this precise with the definition of a *semidirect product*:

Definition.

Given groups N, H, and an action by automorphisms of H on N, that is a homomorphism:

$$\rho : H \to \mathrm{Aut}(N)$$

The *semidirect product* of N with H with respect to ρ is the set $N \times H$ equipped with the following group law:

$$(n_1, h_1)(n_2, h_2) = (n_1\rho(h_1)(n_2), h_1h_2)$$

This is denoted $N \rtimes_\rho H$

Proposition.

$N \rtimes_\rho H$ is a group for every action by automorphisms ρ.

Proof: Before I give the proof, I just want to assure you that in practice you will never have to work through the following calculations explicitly; in practice the semidirect products we deal with are much more explicit, this is just to convince you that the general construction is indeed a group. We verify the existence of identity and inverses and that the group operation is associative, denoting by $\phi_h = \rho(h)$ the action of a particular $h \in H$ – I'll use this abbreviation when the distinction between multiplication and the action can get confusing, but for shorter calculations later on I'll omit it to give you a bit more confidence with the notation $\rho(h)(n)$ since thinking of the output of a function as another function you can evaluate is a good thing to get used to:

1. Identity is given by $e = (e_N, e_H)$. Let $(n, h) \in N \rtimes_\rho H$. Then since automorphisms fix identity:

$$\begin{aligned}(n, h)(e_N, e_H) &= (n\phi_h(e_N), he_H)\\ &= (ne_N, h)\\ &= (n, h)\\ (e_N, e_H)(n, h) &= (e_N\phi_{e_H}(n), e_Hh)\\ &= (e_N\mathrm{id}_N(n), h)\\ &= (e_Nn, h)\\ &= (n, h)\end{aligned}$$

2. Given $(n,h) \in N \rtimes_\rho H$, $(n,h)^{-1} = (\phi_{h^{-1}}(n^{-1}), h^{-1})$. Directly proving:

$$\begin{aligned}
(n,h)(n,h)^{-1} &= (n,h)(\phi_{h^{-1}}(n^{-1}), h^{-1}) \\
&= \left(n\phi_h\left(\phi_{h^{-1}}(n^{-1})\right), hh^{-1}\right) \\
&= (n\phi_{hh^{-1}}(n^{-1}), e_H) \\
&= (n\mathrm{id}_N(n^{-1}), e_H) \\
&= (nn^{-1}, e_H) \\
&= (e_N, e_H) = e_{N\rtimes_\rho H} \\
(n,h)^{-1}(n,h) &= (\phi_{h^{-1}}(n^{-1}), h^{-1})(n,h) \\
&= (\phi_{h^{-1}}(n^{-1})\phi_{h^{-1}}(n), h^{-1}h) \\
&= (\phi_{h^{-1}}(n)^{-1}\phi_{h^{-1}}(n), e_H) \\
&= (e_N, e_H) = e_{N\rtimes_\rho H}
\end{aligned}$$

3. Let $(n_1,h_1),(n_2,h_2),(n_3,h_3) \in N\rtimes_\rho H$. Then, being careful to remember that $\rho(h) = \phi_h$ is always a homomorphism for any $h \in H$ and therefore that $\phi_{h_1}(n_2\phi_{h_2}(n_3)) = \phi_{h_1}(n_2)\phi_{h_1}(\phi_{h_2}(n_3))$, and that since ρ is an action that $\phi_{h_1} \circ \phi_{h_2} = \phi_{h_1h_2}$:

$$\begin{aligned}
((n_1,h_1)(n_2,h_2))(n_3,h_3) &= (n_1\phi_{h_1}(n_2), h_1h_2)(n_3,h_3) \\
&= (n_1\phi_{h_1}(n_2)\phi_{h_1h_2}(n_3), h_1h_2h_3) \\
(n_1,h_1)((n_2,h_2)(n_3,h_3)) &= (n_1,h_1)(n_2\phi_{h_2}(n_3), h_2h_3) \\
&= (n_1\phi_{h_1}(n_2\phi_{h_2}(n_3)), h_1h_2h_3) \\
&= (n_1\phi_{h_1}(n_2)\phi_{h_1}(\phi_{h_2}(n_3)), h_1h_2h_3) \\
&= (n_1\phi_{h_1}(n_2)\phi_{h_1h_2}(n_3), h_1h_2h_3) \\
&= ((n_1,h_1)(n_2,h_2))(n_3,h_3)
\end{aligned}$$

This completes the proof. □

Note.

N and H are naturally embedded as subgroups of $N \rtimes_\rho H$, namely as $N \times \{e\}$ and $\{e\} \times H$. This is because for $n \in N$, $h \in H$ we have that these are closed under the operation and under inversion, and have the same group operations as subgroups as the original groups as N, H:

$$\begin{aligned}
(n,e)^{-1} &= (\rho(e^{-1})n^{-1}, e^{-1}) \\
&= (n^{-1}, e) \\
(n_1,e)(n_2,e) &= (n_1\rho(e)n_2, e^2) \\
&= (n_1n_2, e) \\
(e,h)^{-1} &= (\rho(h^{-1})e^{-1}, h^{-1}) \\
&= (e, h^{-1}) \\
(e,h_1)(e,h_2) &= (e\rho(h_1)(e), h_1h_2) \\
&= (e, h_1h_2)
\end{aligned}$$

However, the projection map $\pi_H : N \rtimes_\rho H \twoheadrightarrow H$ is *not* a homomorphism in general.

So then what we proved up until now has been the following theorem:

Theorem.

If G is a finite group, $N \trianglelefteq G$, $H \leq G$, and every coset of N contains a unique element of H, then we can write every element of G uniquely as nh for $n \in N, h \in H$ and we have an isomorphism:

$$\begin{aligned} N \rtimes_\rho H &\xrightarrow{\sim} G \\ (n, h) &\mapsto nh \end{aligned}$$

For some action by automorphisms $\rho : H \to \mathrm{Aut}(N)$.

Note.

We do actually know what ρ is if we are actually starting from G itself: $\rho(h)(n) = hnh^{-1}$. The usual situation however is that we know only that subgroups of this form exist and we want to classify the possibilities for G given this, which means we want to classify the possibilities for ρ. This is why we phrase the theorem as "for some ρ" rather than saying the particular ρ it is. To convince you that we don't lose any information this way, consider the following calculation:

$$\begin{aligned} (e, h)(n, e)(e, h^{-1}) &= (e\rho(h)(n), h)(e, h^{-1}) \\ &= (\rho(h)(n)\rho(h)(e), hh^{-1}) \\ &= (\rho(h)(n), e) \end{aligned}$$

So indeed in the larger group conjugation always corresponds just to the action of H via ρ.

So this gives us our desired decomposition. I think you'll agree however that one of our conditions is a bit weird. N being normal is fairly natural, H being a subgroup isn't too bad, but it's a bit of a mouthful to say that "every coset of N contains a unique element of H" and a royal pain to verify. Luckily, there's a good equivalent criterion that is equivalent and much more tangible:

Proposition.

Given a subgroups $N, H \leq G$, if $|N||H| = |G|$ and $N \cap H = \{e\}$ then we have that every coset of N contains a unique element of H.

Proof: There is a very quick proof if N is normal: The homomorphism:

$$\begin{aligned} \psi : H &\to G/N \\ h &\mapsto hN \end{aligned}$$

Has kernel $H \cap N = \{e\}$ hence injective and $|H| = \left|G/N\right|$ hence bijective and an isomorphism; for the reasons discussed above this map being an isomorphism is exactly what it means for every coset to contain a unique element of H. Thus $H \cong G/N$ which for the reasons discussed above This is the case we care about most, but the claim is true in general and we can prove it absent the hypothesis of G/N being a group; it just means that instead of showing the kernel is trivial we have to show injectivity directly. Suppose then that $h_1 N = h_2 N$, then there exist n_1, n_2 with $h_1 n_1 = h_2 n_2$ and therefore:

$$n_1 n_2^{-1} = h_1^{-1} h_2 \in H \cap N = \{e\} \implies h_1 = h_2$$

□

So overall our criterion is, adding back the normality condition:

Theorem.

If we have a finite group G, a normal subgroup $N \trianglelefteq G$, and a subgroup $H \leq G$ with $|N||H| = |G|$ and $H \cap N = \{e\}$, then for some $\rho : H \to \text{Aut}(N)$ we have an isomorphism:

$$G \cong N \rtimes_\rho H$$

Note.

When applying this theorem in practice in this class, you will generally have the situation where H is prime order hence generated by some single element $y \in H$. Let $x_1, ..., x_n$ be a generating set of N – in practice this will generally be very small. Then classifying the automorphism actions $\rho : H \to \text{Aut}(N)$ is equivalent to classifying the possible values of $\rho(y)(x_i)$, which since we know that in G this is just yx_iy^{-1} you are just determining what the conjugates of each of the generators is. So what you'll do is classify the automorphisms, then write down what this means in terms of the conjugates of the generators, and then you'll have a full presentation of the group! We'll do this in an example in just a moment.

Before we move on to applications, it's worth quickly noting when this does reduce to an actual direct product. First, a fairly quick Lemma:

Lemma.

If ρ is trivial, that is $\rho(H) = \{\text{id}_N\}$, then $N \rtimes_\rho H = N \times H$.

Proof: The set is trivially the same and the group operation is the same:

$$\begin{aligned}(n_1, h_1)(n_2, h_2) &= (n_1\rho(h_1)(n_2), h_1h_2) \\ &= (n_1\text{id}_N(n_2), h_1h_2) \\ &= (n_1n_2, h_1h_2)\end{aligned}$$

□

Recall now that if $G \cong N \times H$ then it contains subgroups isomorphic to N, H that are normal. We can now prove the converse:

Proposition.

Suppose $N, H \trianglelefteq G$ are both normal subgroups of G with $H \cap N = \{e\}, |H||N| = |G|$. Then $G \cong N \times H$.

Proof: By the theorem, $G \cong N \rtimes_\rho H$ with $\rho(h)(n) = hnh^{-1}$. It therefore suffices to show that $hnh^{-1} = n$ for all $h \in H$ by the preceding lemma. This is equivalent to the two elements commuting, that is $[h, n] = hnh^{-1}n^{-1} = e$. Observe:

$$\overbrace{h\underbrace{nh^{-1}}_{\in H}n^{-1}}^{\in N} \in H \cap N = \{e\}$$

□

One last useful observation:

Lemma.

If $N, H \leq G$ are subgroups, $\gcd(|N|, |H|) = 1$, then $H \cap N = \{e\}$ giving one of the conditions of the above theorem.

Proof: By Lagrange, $|H \cap N| \mid |H|$ and $|H \cap N| \mid |N|$ so $|H \cap N| = 1$. □

3 Applications via Sylow

We're not going to be reviewing the proof of Sylow, because it's not the most vital thing to have on recall and I want to focus on semidirect products. However, it's good to have them here to phrase the applications:

Definition.

If we write $|G| = p^e m$ for $e \in \mathbb{N}$, $p \nmid m$, then a subgroup $H \leq G$ of order p^e is a p-Sylow subgroup of G.

Theorem.

Sylow:

1. For every prime p, a p-Sylow subgroup of G exists.
2. All p-Sylow's are conjugate, and any subgroup of order a power of p is a subgroup of a p-Sylow
3. The number s_p of p-Sylows satisfies $s_p \equiv 1 \mod p$ and $s_p \mid m$

These theorems often allow us to identify candidates for our semidirect product decomposition. As a simple example, let's take a fresh look at the classification of groups of order 6 with an absolutely overkill proof:

Example.

Every group of order 6 is isomorphic to one of $\mathbb{Z}/6\mathbb{Z}$ or S_3

If you're reading this before or during the review session, don't read the proof yet – it's an exercise you can do yourself to check you understand if you're not coming and if you are we'll be going over it in the review session. That said, the proof is on the following page:

Proof.

Suppose $|G| = 6$. Normally at this step in the proof we'd just use the Structure Theorem to eliminate the Abelian case, but I'll treat them all in one to emphasise the process. So, by Sylow there exist a 2-Sylow subgroup H and a 3-Sylow N, with:

$$\begin{aligned} s_3 &\equiv 1 \mod 3 & \qquad s_2 &\equiv 1 \mod 2 \\ s_3 &\mid 2 & \qquad s_2 &\mid 3 \end{aligned}$$

It follows from this that $s_3 = 1$ and hence N is normal[a] and moreover for an arbitrarily chosen 2-Sylow H $H \cap N = \{e\}$ since $\gcd(2,3) = 1$. Therefore since $N \cong \mathbb{Z}/3\mathbb{Z}, H \cong \mathbb{Z}/2\mathbb{Z}$ we have by the above theorem a semidirect product decomposition for some $\rho : \mathbb{Z}/2\mathbb{Z} \to \operatorname{Aut}\left(\mathbb{Z}/3\mathbb{Z}\right)$:

$$G \cong \mathbb{Z}/3\mathbb{Z} \rtimes_\rho \mathbb{Z}/2\mathbb{Z}$$

So we seek to classify the possibilities for $\rho(1) : \mathbb{Z}/3\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/3\mathbb{Z}$. Call $\phi = \rho(1)$. This is moreover determined by $\phi(1)$. If $\phi(1) = 1$ then the action is trivial and in fact we have a direct product:

$$G \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z}$$

So the only nontrivial case is $\phi(1) = 2$. Note that this already gives us the classification if we know that S_3 exists since it shows that there is only one other possible group, but let's finish it off. Let $r \in N$ be a generator and $s \in H$ be a generator, then the statement that $\phi(1) = 2 = -1$ is equivalent to $srs^{-1} = r^{-1}$. Thus:

$$G \cong \langle r, s \mid r^3 = e, s^2 = e, srs = r^{-1} \rangle$$

But this is exactly the presentation of $D_3 \cong S_3$. □

[a] conjugates of p-sylows are p-sylows and hence if there is only 1 p-sylows it is its own conjugate and hence it is normal

Note.

Note that since 1 has order 2 in $\mathbb{Z}/2\mathbb{Z}$ $\phi = \rho(1)$ must also have order 2, that is $\phi^2 = \rho(0) = \mathrm{id}$. This is trivially true in this case but important to watch out for in general.

One last thing: Auroux didn't really have time to carefully go through the last part of the classification of groups of order 12, where the 3-Sylow is normal but the 2-Sylow isn't. So let's take a look at that:

Example.

Classify groups of order 12 where the 3-Sylow is normal but the 2-Sylow isn't.

Proof.

Let $N = \langle x \rangle \cong \mathbb{Z}/3\mathbb{Z}$ be the 3-Sylow and H the 2-Sylow, $|H| = 4$. First suppose $H = \langle y \rangle \cong \mathbb{Z}/4\mathbb{Z}$. Then we know that we have a semidirect product:

$$G \cong N \rtimes_\rho H$$

For some automorphism action:

$$\rho : \mathbb{Z}/4\mathbb{Z} \to \operatorname{Aut}\left(\mathbb{Z}/3\mathbb{Z}\right)$$

Recall that this is entirely determined by $\rho(1)(1)$, since H is cyclic. If $\rho(1)(1) = 1$ then the action is

trivial which cannot happen since by hypothesis H is not normal which it would be in a direct product. Therefore $\rho(1)(1)$ must be the only other automorphism of $\mathbb{Z}/3\mathbb{Z}$, $\rho(1)(1) = 2 = -1$. Now remember that $\rho(1)(1)$ is exactly the power of the generator of N that yxy^{-1} maps to. This then means that $yxy^{-1} = x^{-1}$ and we have the following presentation:

$$G \cong \langle x, y \mid x^3 = y^4 = 1, yxy^{-1} = x^{-1} \rangle$$

This group isn't one we know, but we do have notation for it. Notice that this is the unique nontrivial semidirect product of $\mathbb{Z}/3\mathbb{Z}$ with $\mathbb{Z}/4\mathbb{Z}$, the only other one coming from the trivial action which gives the direct product, and as such is uniquely specified and we can just write without a subscript:

$$G \cong \mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$$

Now suppose $H = \langle y, z \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We seek to classify automorphism actions:

$$\rho : H \to \mathrm{Aut}\,(N)$$

Now note that $\mathrm{Aut}(N) \cong \mathrm{Aut}\left(\mathbb{Z}/3\mathbb{Z}\right) \cong \mathbb{Z}/2\mathbb{Z}$. By first isomorphism theorem we must have that $\ker \rho$ is a nontrivial subgroup of $H \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Call y the generator of $\ker \rho$, then $yxy^{-1} = x$. Now since $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \mathbb{F}_2^2$ we can complete this to a generating set and write $H = \langle y, z \rangle$, and for the action to be nontrivial we must have $\rho(z) \neq x$, that is if we write things in coordinates using z as our second generator that $\rho(0,1)(1) \neq 1$. Therefore our only other option is $zxz^{-1} = x^{-1}$. This then fixes a presentation for our group:

$$G \cong \langle x, y, z \mid x^3 = y^2 = z^2 = e, yxy^{-1} = x, zxz^{-1} \rangle$$

This action implies that $G \cong D_6$. This is because we have the following isomorphism:

$$\begin{aligned} \psi : G &\xrightarrow{\sim} D_6 \\ x &\mapsto r^2 \\ y &\mapsto r^3 \\ z &\mapsto s \end{aligned}$$

So indeed:

$$G \cong D_6$$

□

Note.

There was an important bit of sleight of hand there. Notice that there are a couple different nontrivial automorphism actions of $H \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on $N \cong \mathbb{Z}/3\mathbb{Z}$. Normally what we would have done would have been to specify generators of each and then analyze how each of the generators would have acted. This would have given you several seemingly different answers. These would have however been the same up to relabelling. So instead we reasoned about what properties the automorphism has and chose our generators judiciously to respect that. This reflects a general principle that semidirect products with respect to automorphism actions that differ only by an automorphism – think relabelling – of one of the faktors of the semidirect product are isomorphic.