MATH 55b: Studies in Real and Complex Analysis — Review Session

Homotopies and the Fundamental Group

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Brief Recap: Urysohn's Lemma and the Quotient Topology

Theorem 0.1 (Urysohn Metrisation Theorem)**.** *Let X be a topological space that is* regular[1] *and has a countable basis. Then X is metrizable; i.e. there exists a metric whose induced topology coincides with the given topology on X.*

Lemma 0.2 (Urysohn's Lemma)**.** *Let X be a* normal *topological space and let $A, B \subset X$ be disjoint closed subsets. Then there exists a continuous function $f\colon X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

Claim 0.3 (Countable separating family)**.** There exists a countable family of continuous maps $\{f_n\}_{n\geq 1}$ with $f_n\colon X \to [0,1]$ such that for every $x \in X$ and every open neighbourhood $U \ni x$ there is some index n satisfying $f_n(x) = 1$ and $f_n \equiv 0$ on $X \setminus U$.

Proposition 0.4 (Embedding into a cube)**.** *With the family $\{f_n\}$ from Claim 0.3, the map*

$$F\colon X \longrightarrow [0,1]^{\mathbb{N}}, \qquad x \longmapsto \big(f_1(x), f_2(x), \dots\big)$$

is an embedding*; that is, F is continuous, injective, and a homeomorphism from X onto its image $F(X)$ endowed with the subspace topology.*

Proposition 0.5 (Metrisability of the countable product cube)**.** *The countable product $[0,1]^{\mathbb{N}}$ equipped with the product topology is metrizable. One metric inducing this topology is*

$$d\big((x_n)_{n\geq 1}, (y_n)_{n\geq 1}\big) = \sup_{n\geq 1} \frac{|x_n - y_n|}{n}.$$

Definition 0.6 (Quotient topology)**.** Let $q\colon X \twoheadrightarrow A$ be a surjective map from a topological space X to a set A. The *quotient topology* on A is defined by declaring a set $U \subset A$ to be open whenever $q^{-1}(U)$ is open in X.

Definition 0.7 (Quotient map)**.** A surjective continuous map $f\colon X \to Y$ is a *quotient map* if a subset $U \subset Y$ is open whenever $f^{-1}(U)$ is open in X. Equivalently, Y carries the quotient topology induced by f.

Theorem 0.8 (Universal property of quotient spaces)**.** *Let $q\colon X \twoheadrightarrow X/\sim$ be the canonical projection onto a quotient space. For any topological space Y and any function $f\colon X \to Y$ that is constant on equivalence classes (i.e. $x \sim x' \Rightarrow f(x) = f(x')$), the induced map*

$$\bar{f}\colon X/\sim \longrightarrow Y, \qquad \bar{f}([x]) = f(x),$$

is continuous. Moreover, $\bar{f}$ is the unique continuous map satisfying $\bar{f} \circ q = f$.

[1] Every point can be separated from every closed set by disjoint open neighbourhoods.

1 Homotopy and the Fundamental Group

Definition 1.1 (Homotopy)**.** For continuous maps $f, g\colon X \to Y$, a *homotopy* from f to g is a continuous map $H\colon X \times I \to Y$ (where $I = [0,1]$) such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. If such an H exists we write $f \simeq g$ and say that f and g are *homotopic*. A map that is homotopic to a constant map is called *nullhomotopic*.

Definition 1.2 (Path and path homotopy)**.** A *path* in X from x_0 to x_1 is a continuous map $f\colon I \to X$ with $f(0) = x_0$ and $f(1) = x_1$. Two such paths f, g are *path–homotopic*, written $f \simeq_p g$, if there exists a homotopy $H\colon I \times I \to X$ fixing the end–points, i.e. $H(s,0) = f(s)$, $H(s,1) = g(s)$ and $H(0,t) = x_0$, $H(1,t) = x_1$ for all $s, t \in I$.

Lemma 1.3. *Homotopy ($\simeq$) and path homotopy ($\simeq_p$) are equivalence relations.*

Definition 1.4 (Path concatenation)**.** Given paths $f\colon I \to X$ from x to y and $g\colon I \to X$ from y to z, their *concatenation* $f * g\colon I \to X$ is defined by

$$(f * g)(s) = \begin{cases} f(2s), & 0 \le s \le \frac{1}{2}, \\ g(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$

Claim 1.5 (Groupoid structure)**.** Concatenation descends to path–homotopy classes and is

(i) associative;

(ii) admits identity morphisms given by constant paths; and

(iii) every path class $[f]$ has an inverse $[\bar{f}]$, where $\bar{f}(s) = f(1-s)$.

Hence the collection of path–homotopy classes of paths in X forms the *fundamental groupoid* $\Pi_1(X)$.

Definition 1.6 (Fundamental group)**.** Fix a base point $x_0 \in X$. The *fundamental group* of X at x_0 is

$$\pi_1(X, x_0) = \Pi_1(X)(x_0, x_0),$$

the set of path–homotopy classes of loops based at x_0, with multiplication given by concatenation.

Definition 1.7 (Simply–connected space)**.** A topological space X is *simply–connected* if it is path–connected and its fundamental group is trivial, i.e. $\pi_1(X, x) = \{1\}$ for some (and hence every) $x \in X$.

Definition 1.8 (Homotopy equivalence)**.** Spaces X and Y are *homotopy equivalent* if there exist maps $f\colon X \to Y$ and $g\colon Y \to X$ with $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

Definition 1.9. Let $A \subset X$.

(i) A *retraction* is a map $r\colon X \to A$ with $r|_A = \mathrm{id}_A$.

(ii) A is a *deformation retract* of X if there is a homotopy $H\colon X \times I \to X$ from id_X to the inclusion $i\colon A \hookrightarrow X$ that fixes A pointwise.

Proposition 1.10. *If A is a deformation retract of X, then A and X are homotopy equivalent; in particular they have isomorphic fundamental groups.*

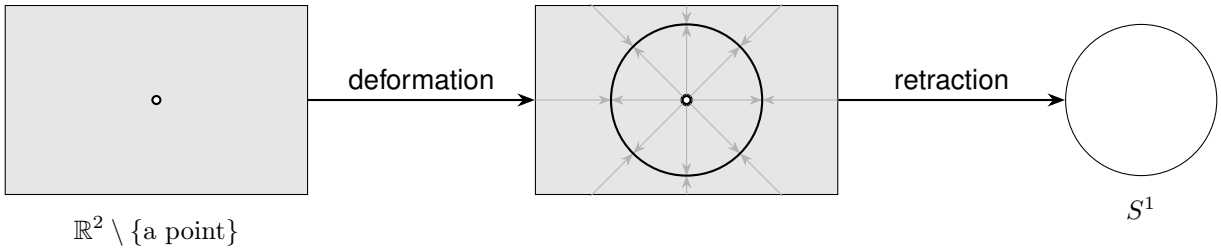


Figure 1: Deformation retract of $\mathbb{R}^2 \setminus \{\text{a point}\}$ onto a circle, implying $\mathbb{R}^2 \setminus \{\text{a point}\}$ and S^1 have isomorphic fundamental groups by Proposition 1.10.

2 Properties of π_1

Proposition 2.1. *Let $x_0, x_1 \in X$ lie in the same path component and let α be a path from x_0 to x_1. The map*

$$\alpha_* \colon \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1), \qquad [\gamma] \mapsto [\bar{\alpha} * \gamma * \alpha],$$

is a group isomorphism. Hence the fundamental groups at any two points in a path–connected space are isomorphic (though not canonically).

Corollary 2.2. *If X is path–connected, the isomorphism class of $\pi_1(X, x)$ is independent of the choice of base point x. Moreover, for any loop ℓ at x_0, conjugation by $[\ell]$ defines an* inner automorphism *of $\pi_1(X, x_0)$.*

Definition 2.3. Given a continuous map of pointed spaces $h \colon (X, x_0) \to (Y, y_0)$, the induced homomorphism

$$h_* \colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0), \qquad h_*([f]) = [h \circ f],$$

is called the *fundamental group homomorphism* determined by h.

Proposition 2.4 (Functoriality of π_1)**.** *For pointed maps $(X, x_0) \xrightarrow{h} (Y, y_0)$ and $(Y, y_0) \xrightarrow{k} (Z, z_0)$ we have $(k \circ h)_* = k_* \circ h_*$. Hence*

$$\pi_1 \colon (\textit{pointed topological spaces}) \longrightarrow (\textit{groups})$$

is a covariant functor.

Theorem 2.5 (Homotopy Equivalences and π_1)**.** *If $f \colon (X, x_0) \to (Y, y_0)$ is a homotopy equivalence (with homotopy inverse g), then the induced map $f_* \colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.*

Proposition 2.6 (Independence of Homotopic Maps)**.**

(1) If $h, k \colon X \to Y$ are homotopic through a homotopy H that fixes *the base point y_0, then $h_* = k_*$.*

(2) If H does not fix base points, let $\alpha(t) = H(x_0, t)$. Then h_* and k_* are related by the isomorphism α_* of Proposition 2.1: $k_* = \alpha_* \circ h_* \circ \alpha_*^{-1}$.

Definition 2.7 (Covering Map)**.** A continuous surjective map $p \colon E \twoheadrightarrow B$ is a *covering map* if every point $b \in B$ possesses an open neighbourhood U such that $p^{-1}(U) = \bigsqcup_{\alpha \in A} V_\alpha$, where each V_α is open in E and $p|_{V_\alpha} \colon V_\alpha \to U$ is a homeomorphism. The sets V_α are called *sheets* or *slices*, and E is a *covering space* of B.

Theorem 2.8 (Product Coverings)**.** *If $p \colon E \to B$ and $q \colon E' \to B'$ are covering maps, then the product map*

$$p \times q \colon E \times E' \longrightarrow B \times B', \qquad (e, e') \longmapsto (p(e), q(e')),$$

is also a covering map.

3 Lifting Properties

Definition 3.1 (Lift of a Map)**.** Given a covering map $p\colon E \to B$ and a continuous map $f\colon X \to B$, a *lift* of f is a map $\widetilde{f}\colon X \to E$ satisfying $p \circ \widetilde{f} = f$.

Theorem 3.2 (Path Lifting)**.** *Let $p\colon E \to B$ be a covering map and $f\colon [0,1] \to B$ a path with $f(0) = b$. For every $e \in p^{-1}(b)$ there exists a unique lift $\widetilde{f}\colon [0,1] \to E$ such that $\widetilde{f}(0) = e$.*

Theorem 3.3 (Homotopy (Square) Lifting)**.** *Let $F\colon I \times I \to B$ be continuous with $F(0,0) = b$ and let $e \in p^{-1}(b)$. Then there exists a unique lift $\widetilde{F}\colon I \times I \to E$ with $\widetilde{F}(0,0) = e$.*

Remark 3.4 (Lifting Correspondence)**.** For a covering $(E, e_0) \xrightarrow{p} (B, b_0)$, the *lifting correspondence*

$$\varphi\colon \pi_1(B, b_0) \longrightarrow p^{-1}(b_0), \qquad \varphi([f]) = \widetilde{f}(1),$$

associates to each loop class the end–point of its lift starting at e_0.

Proposition 3.5 (Surjectivity of φ)**.** *If the covering space E is path–connected, the lifting correspondence φ is surjective.*

Theorem 3.6 (Simply–Connected Cover)**.** *If $p\colon (E, e_0) \to (B, b_0)$ is a covering and E is simply connected, then*

$$\varphi\colon \pi_1(B, b_0) \longrightarrow p^{-1}(b_0)$$

is a bijection.

Example 3.7. The fundamental group of the circle is isomorphic to the integers:

$$\pi_1(S^1) \cong \mathbb{Z}.$$

seen by taking the covering map $p : (\mathbb{R}, 0) \to (S^1, (1,0)), \quad x \mapsto (\cos 2\pi x, \sin 2\pi x)$, which by Theorem 3.6 gives that $\varphi : \pi_1(S^1, (1,0)) \to p^{-1}((1,0)) = \mathbb{Z}$ is a bijection. Extending this inspection, by Theorem 2.8 we have that the fundamental group of the torus $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$.

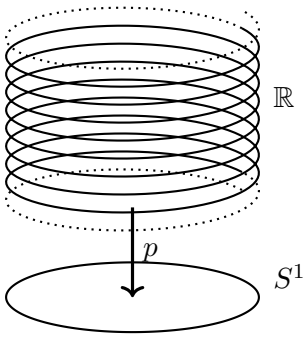


Figure 2: A visualization of our covering map p, "wrapping" length-1 intervals of $\mathbb{R}$ around S^1.

Practice Problems

Given spaces X and Y , let $[X, Y]$ denote the set of homotopy classes of maps of X into Y. For $I = [0, 1]$, show that for any X, the set $[X, I]$ has a single element, and if Y is path-connected then $[I, Y]$ has a single element too.

Let x_0 and x_1 be points of the path-connected space X. Show that $\pi_1(X, x_0)$ is abelian if and only if for every pair (α, β) of paths from x_0 to x_1, we have $\alpha_* = \beta_*$.

Show that if X is path–connected, then the homomorphism on fundamental groups induced by a continuous map is independent of the choice of base-point, up to the canonical change–of–base–point isomorphisms. More precisely, let $h : X \to Y$ be continuous with $h(x_0) = y_0$ and $h(x_1) = y_1$. Let $\alpha : I \to X$ be any path from x_0 to x_1, and set $\beta = h \circ \alpha$. show that

$$\beta_* \circ (h_{x_0})_* = (h_{x_1})_* \circ \alpha_*$$

so that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{(h_{x_0})_*} & \pi_1(Y, y_0) \\ \downarrow \alpha_* & & \downarrow \beta_* \\ \pi_1(X, x_1) & \xrightarrow{(h_{x_1})_*} & \pi_1(Y, y_1) \end{array}$$

Show that $\mathbb{R}^3 \setminus \{\text{a straight line}\}$ deformation retracts to a circle, and find its fundamental group. How about for $\mathbb{R}^2 \setminus \{\text{two points}\}$? What about $\mathbb{R}^2 \setminus \{\text{a line}\}$?