

Math 55b Week 1 Review Session

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1 Metric and Topological Spaces

A topological space is a lot like a group, vector space, or representation. It is an incredibly useful mathematical object that so much can be said about. Unlike those examples you saw in 55a, though, it is not an algebraic object—it is not given a structure by an operation, but rather a designation for subsets, the *open sets*. Also, much like groups/vector spaces/representations, there are some more intuitive examples of the general phenomena that let us get a handle on why they're so powerful.

Definition 1.1. A *metric space* (X, d) is a set X together with a distance function d , such that for all $p, q, r \in X$,

- $d(p, q) = 0 \Leftrightarrow p = q$.
- $d(p, q) = d(q, p)$.
- $d(p, r) \leq d(p, q) + d(q, r)$ (The Triangle Inequality!)

Metric spaces represent the notion of any space with distance. This can be normal space with the normal Euclidean distance, or it can be some sort of weird alien function that fits those rules. One of my favorite examples of a weird distance is the Taxicab metric, which defines $d = \sum |x_i - y_i|$ (it is called the Taxicab metric because it measures displacement in each of the coordinate axis, which is the distance a cab has to travel to get around say New York City).

Definition 1.2. Given any metric space (X, d) , and any $x \in X$ and $r \in \mathbb{R}_{\geq 0}$, we have the notion of *the open ball of radius r around x* given by $B_r(x) := \{y \in X \mid d(x, y) < r\}$.

Definition 1.3. In general, $U \subset X$ is open if for all $p \in U$, there is some open ball around p contained in U .

Definition 1.4. A map $f : X \rightarrow Y$ between metric spaces is continuous if for all $p \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $d_X(p, x) < \delta \Rightarrow d_Y(f(p), f(x)) < \epsilon$.

Now, metric spaces are just a fancy example of a general topological space. You will prove in a mixture of lecture and homework (I think) that the following definitions are equivalent to those above in the special case of a metric space, but they also capture many more phenomena that can't be explained in terms of distance.

Definition 1.5. A topological space is a set X and a subset $\tau \in P(X)$ which we call the open sets of X , such that $\emptyset \in \tau$, any arbitrary union of elements of τ is in τ , and any finite intersection of elements in τ is also in τ .

Definition 1.6. A map $f : X \rightarrow Y$ between topological spaces is continuous if every open $U \subset Y$ has open preimage in X .

In either scheme, we say that a set is closed if its complement is open. Sets can be neither closed or open, or they can be both (clopen). The finite union and arbitrary intersection of closed sets are closed. There are a couple of terms relating open and closed sets that are crucial to know:

Definition 1.7. Given any $A \subset X$, we have

- The *closure* of A , written $\bar{A}$, is the smallest closed set containing A .
- The *interior* of A , written $\text{int}(A)$, is the largest open set contained in A .
- The *boundary* of A , written ∂A , is $\bar{A} - \text{int}(A)$.
- We say that A is the *neighborhood* of some $p \in X$ if A is open and $p \in A$.
- We say that A is *dense* if $\bar{A} = X$.

Lastly, we have the notion of a structure-preserving map between topological spaces:

Definition 1.8. A map $f : X \rightarrow Y$ is a *homeomorphism* if it is a continuous bijection whose inverse is also continuous.

Note that the inverse might not always be continuous, so this is a necessary part of the definition, but it is sufficient to check that f is also a bijection of open sets.

2 Limit Points and Sequences

Limit points and sequences matter in all of topology, but there are some definitions in metric spaces which don't transfer to topological spaces.

Definition 2.1. A sequence $\{p_n\}$ converges to a limit p if every neighborhood $U \ni p$ contains every p_n except for finitely many n . In a metric space, this is equivalent to that for all $\epsilon > 0$, there exists an $N > 0$ such that $d(p_n, p) < \epsilon$ for all $n \geq N$.

Definition 2.2. A sequence $\{p_n\}$ is *Cauchy* if for every $\epsilon > 0$, there exists an $N > 0$ such that $d(p_m, p_n) < \epsilon$ for all $m, n > N$.

Only sequences in a metric space can be Cauchy. In this case, any convergent sequence is Cauchy. It is not necessary in every space that every Cauchy sequence is convergent, if this is true of a space, we say that it is *complete*.

We also have the notion of limit points of sets.

Definition 2.3. A point $x \in X$ is a limit point of $A \subset X$ (x itself may or may not lie in A) if for every neighborhood $U \ni x$, it is true that $A \cap (U - \{x\}) \neq \emptyset$.

How does this connect? Well, it can be shown that the closure of A is A together with all of its limit points, and that any sequence contained in A converges a point in $\bar{A}$, so either a point in A or one of its limit points.

3 Ways to get Topologies

The easiest way to get a topology is to give a *basis* for this topology. Much like a basis for a vector space, we can view a basis for a topology as a set of building blocks for the important elements (just open sets instead of vectors). Instead of taking vectors that can make any other vector through linear combinations, we take open sets that can make any other open set through arbitrary union.

Definition 3.1. If $\mathcal{B} \subset P(X)$ is such that $\bigcup_{B \in \mathcal{B}} B = X$ and $x \in B_1 \cap B_2$, then there is some $B' \in \mathcal{B}$ such that $x \in B' \subset B_1 \cap B_2$, then every open set in X can be given by the union of elements in $\mathcal{B}$. We say that $\mathcal{B}$ is a basis and that it generates the topology of X .

We can also get a new topology by restricting a topology to a subspace, or by taking the product of two topologies.

Definition 3.2. The subspace topology on $Y \subset X$ has open sets $U \cap Y$ for all open $U \subset X$.

Definition 3.3. The product topology on $X \times Y$ has open sets $U \times V$ for all open $U \subset X$ and $V \subset Y$.

If we have multiple topologies on a set X , we can say one is *finer* than the other if it has all the same open sets and more. One is *coarser* than the other if it has some of the open sets, but not all, and no more.

4 Proofwriting

The goal of a proof is to explain logically why a claim must be true, from previous results, in a logically infallible and rigorous way. Producing a rigorous proof confirms that a statement is true, as long as all the assumptions used have been proved as well. Proofs also illuminate the underlying reasons why exactly a certain pattern or condition always hold.

There are a couple of techniques you are going to see in 55:

- **Direct Proof**

A proof that follows a direct chain of reasoning is known as a direct proof. This is likely the most common type of proof you will use. In general, using a direct proof can be more illuminative and clear than the techniques described below, although they all have their place in rigorous mathematical proofs.

Claim 4.1. *The square of an even number is even.*

Proof. Let n be even. This means we can write $n = 2k$ for some $k \in \mathbb{N}$. Then, $n^2 = (2k)^2 = 4k^2$ is a multiple of four, and thus even. $\square$

- **Proof By Contradiction**

If you are trying to show that some statement is true, it works just as well to show that the alternative is false. Proof by contradiction involves taking the direct opposite of a statement, and showing that it logically implies something which cannot hold.

Claim 4.2. *The square root of two is irrational.*

Proof. For the sake of contradiction, let the square root of two be rational. Then, there exist coprime $p, q \in \mathbb{Z}$ such that $\sqrt{2} = \frac{p}{q}$, or equivalently, $2q^2 = p^2$. This means p^2 is even, so p must also be even. We can write $p = 2r$, for some $r \in \mathbb{Z}$. Then, $2q^2 = (2r)^2 = 4r^2$, so $q^2 = 2r^2$, meaning q^2 and thus q are even. However, p and q both being even contradicts them sharing no factors. $\square$

- **Proof By Induction**

Proof by Induction is a strategy that is often compared to climbing up the rungs of the ladder. First, you prove the statement for the base case, typically $n = 1$ or $n = 2$. Then, you assume the inductive hypothesis, which is that the statement holds up to some n , and with that prove it works for $n + 1$ (the inductive case). Together, these two cases prove that the statement holds for all n .

You can either use induction on a fact that is directly about the natural numbers, or on objects which can be labeled by a natural number. For example, you can prove facts about a group by inducting on the number of elements, or prove facts about graphs by inducting on the number of vertices.

Claim 4.3. $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

Proof. We proceed by induction on n . For $n = 1$, observe that $1 = \frac{1 \cdot 2}{2}$. Now, if $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for some n , then

$$1 + 2 + \cdots + n + (n + 1) = \frac{n(n + 1)}{2} + n + 1 = \frac{n^2 + n + 2n + 2}{2} = \frac{(n + 1)(n + 2)}{2},$$

which concludes the inductive case. $\square$

- **Proof By Contrapositive**

The contrapositive of the statement "if A then B" is "if not B, then not A." This is logically equivalent to the original claim (why?). Make sure not to confuse the contrapositive with the converse, which is "if B, then A."

Claim 4.4. *If ab is odd for integers a, b , then both a, b are odd.*

Proof. We shall prove that if one of a, b is even, then ab is even. Without loss of generality, let a be even. This means we can write $a = 2c$, for some integer c . Then, $ab = 2bc$ is even. $\square$

These are some common errors that can often come up when beginning your proofwriting journey.

- **Lack of Rigor**

Every step of your proof should be justified, either by a definition, a previous result, or valid manipulation. This makes sure that there are no flaws in the argument, and helps make your line of reasoning more clear.

- **Handwaving**

Saying "it can be shown" or "one can check" may make sense for explaining a line of reasoning to a friend, but in a written proof, you need to actually show or check your claim. Make sure to explicitly show algebraic manipulations and casework. Also, use formal induction rather than ellipses to show an argument continuing.

- **Lack of Specificity**

Make sure to label variables ("there exists such an x ") rather than referring to "some number." Also, name the theorems you use ("by the first isomorphism theorem"). This does not have to be done for every single thing you reference, but it is better to err on the side of more specificity.

- **Overcomplication**

If there are two valid ways of proving a claim, the shorter one is the better proof. You won't lose points in 55 for making a longer proof, but it is good practice to try to make your arguments simpler. This often involves using strong (very applicable and powerful) theorems rather than brute calculations or starting from first principles. Simplifying proofs can also be a massive timesaver for you.

Here are some tips for proofwriting in 55, which have helped me (Lillian) immensely.

- **Utilize Course Notes**

Luckily for you, Math 55 is a very self-contained course, so there is very little you will have to reference from outside of what is explained in lecture. Use this to your favor. If you are unsure how to approach a problem about a certain concept, look back at the Lecture Notes or your own personal notes on that concept. Chances are, you'll find the key theorem or fact for your proof.

- **Start With Definitions**

It's very difficult to prove a statement that you don't understand. Making sure you understand the definitions of everything involved with the problem makes sure you are equipped to tackle it. Also, since the conclusion of a proof must follow directly from definitions and previous theorems, figuring out what the definitions imply gets you halfway towards solving the problem.

- **Prioritize Clarity**

You will be writing complex proofs that require many steps, so you should always make sure that you fully understand every step, and that your explanation fully conveys your thoughts. If your solution couldn't fully convince someone else that your reasoning is correct if read aloud, it probably needs to be more clear.

- **Take a Walk**

Many of the problems you will see cannot be solved in a single go-through. Make sure to take a step back from a problem if you have been dedicating too much time to it, and let yourself come back to it with a clear mind.

- **Collaborate!**

Working with other people is massively helpful for tackling problems, and learning how to communicate your mathematical ideas aloud also makes writing proofs easier.