55B Week 6 Review Notes

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December 5th, 2025

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1 Homotopy on the Circle and Brouwer Fixed Point

1.1 Brouwer Fixed Point Theorem

Definition.

- $$B^n = \overline{B_1(0)} = \left\{ (x_1, \ldots, x_n) \mid \sum_{i=1}^{n} x_i^2 \leq 1 \right\} \subset \mathbb{R}^n$$
- $$S^n = \partial B^{n+1} = \left\{ (x_1, \ldots, x_{n+1}) \mid \sum_{i=1}^{n+1} x_i^2 = 1 \right\} \subset \mathbb{R}^{n+1}$$

Note.

∂M indicates the boundary of a space M; notices that the superscript in B^n matches the dimension of euclidean space that it's most naturally embedded in but doesn't for S^n; this is because the superscript is meant to indicate the *manifold dimension* of the space. Don't worry if that doesn't fully make sense to you; informally it just means that if you look very closely at a neighbourhood of a typical point it looks like n-dimensional space

Also recall:

Definition.

A *retraction* $r : X \twoheadrightarrow A$ with $A \subset X$ is a surjective continuous map that fixes A pointwise, that is $r(a) = a$ for every $a \in A$.

Proposition.

Any retraction $r : X \twoheadrightarrow A$ induces a surjective group homomorphism $r_* : \pi_1(X, a_0) \twoheadrightarrow \pi_1(A, a_0)$

Theorem.

There is no retraction $B^2 \twoheadrightarrow S^1$

Proof.

Such a map r has the property that, if $i : S^1 \hookrightarrow B^2$ is the inclusion of S^1 into B^2, then $r \circ i = \mathrm{id}_{S^1}$ – this is just a rephrasing of the fact that r fixes S^1 pointwise. then that means we may factor the identity map $\mathrm{id}_{S^1} : S^1 \to S^1$ as $r \circ i$, meaning we have a diagram:

$$\begin{tikzcd} S^1 \arrow[r, hook, "i"] \arrow[rr, bend right, "\mathrm{id}_{S^1}"'] & B^2 \arrow[r, two heads, "r"] & S^1 \end{tikzcd}$$

Functors preserve diagrams, that is $(\mathrm{id}_{S^1})_* = (r \circ i)_* = r_* \circ i_*$, remembering that the subscript $*$ indicates the induced map on fundamental groups.[a] So we therefore have the following diagram:

$$\begin{tikzcd} \pi_1(S^1, s_0) \arrow[r, hook, "i_*"] \arrow[rr, bend right, "\mathrm{id}_{\pi_1(S^1, s_0)}"'] & \pi_1(B^2, s_0) \arrow[r, two heads, "r_*"] & \pi_1(S^1, s_0) \end{tikzcd}$$

But B^2 is simply connected! that means that the middle node of this diagram is actually the trivial group $\{e\}$. But any homomorphism that goes through a trivial group must be trivial, meaning we have shown that the identity function on $\pi_1(S^1)$ is trivial; but as we will see in the covering space section, $\pi_1(S^1) \cong \mathbb{Z}$ and as such this cannot be the case. ■

[a] $r_*([\gamma]) = [r \circ \gamma]$. In a general setting, this construction of getting an induced maps on spaces of functions by postcomposition is called a *pushforward* construction; you do however have to be careful about this terminology as it does also have another category-theoretic meaning

Corollary. Brouwer Fixed Point Theorem

Every map $B^2 \to B^2$ has a fixed point

Proof.

A map $B^2 \to B^2$ lacking fixed points would allow us to define a retraction $B^2 \twoheadrightarrow S^1$ via mapping $x \in B^2$ to the intersection point of the ray from x through $f(x)$ with S^1, contradicting the previous theorem. ■

1.2 Homotopy on the Circle

Because $S^1 \cong {}^{[0,1]}\!/_{0 \sim 1}$, a loop in (X, x_0) can equivalently be defined as a map $(S^1, (1,0)) \to (X, x_0)$. Thus the behaviour up to homotopy of maps out of S^1 is of great interest to us. The key fact is:

Lemma.

For $h : S^1 \to X$, the following are equivalent:

1. h is nulhomotopic
2. h extends to a continuous map $k : B^2 \to X$,that is $k_{|\partial B^2} = h$.
3. $h_* : \pi_1(S^1, s_0) \to \pi_1(X, x_0)$ is the trivial homomorphism

Proof.

1. (1) $\Longrightarrow$ (2): Notice that the map

$$\begin{aligned} S^1 \times I &\xrightarrow{p} B^2 \\ (x,t) &\mapsto tx \end{aligned}$$

Is a quotient map, meaning that:

$$B^2 \cong {}^{S^1 \times I}\!/_{(x,0) \sim (x',0) \forall x, x' \in S^1}$$

This reflects the fact that gluing the bottom circle of a cylinder all to one point gets you a circle. If I have a nulhomotopy $H : S^1 \times I \to X$ for some constant map $\mathbf{c}$, this says exactly that $H(x,0) = H(x',0) = c$ for all $x, x' \in S^1$. That means it factors through the quotient space, which is B^2! It then suffices to observe that p viewed as a map $B^2 \to B^2$ fixes S^1.

2. (2) $\Rightarrow$ (3): If $h = k|_{S^1}$, then we can write

$$h = k \circ i,$$

where $i : S^1 \hookrightarrow B^2$ is the inclusion. By functoriality of π_1,

$$h_* = k_* \circ i_*,$$

i.e. we have a commutative diagram

$$\pi_1(S^1) \xrightarrow{i_*} \pi_1(B^2) \xrightarrow{k_*} \pi_1(X). \qquad (h_* : \pi_1(S^1) \to \pi_1(X))$$

But $\pi_1(B^2) = \{1\}$, so k_* is trivial, hence h_* is trivial as well.

3. (3) $\Rightarrow$ (1): Assume $h_* : \pi_1(S^1) \to \pi_1(X)$ is trivial. Then the loop

$$f : I \to X, \qquad s \longmapsto h\big(e^{2\pi i s}\big)$$

($= h\circ$ the standard loop going around S^1) represents the trivial element of $\pi_1(X, x_0)$, where $x_0 = h(1)$. Hence there exists a path-homotopy

$$F : I \times I \to X$$

from f to the constant loop at x_0. Note that

$$F(0,t) = F(1,t) = x_0 \qquad \forall\, t \in I.$$

This implies that F factors through the quotient

$$I \times I \big/ \big((0,t) \sim (1,t)\ \forall\, t\big),$$

which is homeomorphic to $S^1 \times I$. Therefore F is in fact a nulhomotopy in S^1. ■

1.3 Fundamental Theorem of Algebra

We can also use this theory to prove the Fundamental Theorem of Algebra! This won't be important in this class but is a really fun fact, although there do exist better proofs.[1]

Theorem. Fundamental Theorem of Algebra

Every polynomial $p(z) \in \mathbb{C}[z]$ has a root

Proof.

For $|z| = R \gg 0$, the term z^d dominates (as soon as $R^k > d \cdot |a_{d-k}|$ for all$1 \le k \le d$), so that

$$\left|a_{d-k} z^{\,d-k}\right| < \frac{1}{d}\, R^d.$$

Hence the straight line segment from $f(z)$ to z^d does not cross 0, . Equivalently,

$$F(z,t) := (1-t) f(z) + t z^d$$

has no zeros on $\{z \in \mathbb{C} : |z| = R\} \times I$. Therefore the maps $S^1 \to \mathbb{C}^*$ defined by

$$e^{i\theta} \longmapsto f\big(Re^{i\theta}\big) \qquad \text{and} \qquad e^{i\theta} \longmapsto R^d e^{id\theta}$$

are homotopic via

$$(e^{i\theta}, t) \longmapsto F\big(Re^{i\theta}, t\big)\,.$$

Hence they induce the same homomorphism

$$\pi_1(S^1) \cong \mathbb{Z} \longrightarrow \pi_1(\mathbb{C} \setminus \{0\}) \cong \mathbb{Z},$$

[1] Come back and ask me about this once we've covered the Open Mapping Principle in Complex Analysis

which is nontrivial, as can be seen from the second map (it sends the generator $1 \in \mathbb{Z}$ to $d \in \mathbb{Z}_{>0}$). Thus the map $e^{i\theta} \mapsto f(Re^{i\theta})$ does not extend to a map $B^2 \to \mathbb{C} \setminus \{0\}$. But if f had no roots, then $z \mapsto f(Rz)$ would provide such an extension. ■

2 Covering Spaces

2.1 Preliminaries

Definition.

Let $p : E \to B$ be a continuous surjection. An open subset $U \subset B$ is *evenly covered* if we may write[a]

$$p^{-1}(U) = \coprod_{\alpha \in A} V_\alpha$$

Where the V_α are disjoint opens with the property that $p_{|V_\alpha} : V_\alpha \xrightarrow{\sim} U$ is a homeomorphism for each $\alpha \in A$. We call the V_α "slices" or "sheets"

[a] $\coprod$ is in this context the same notation as $\bigsqcup$, I just prefer it as it situates it better in a general categorical context

Note.

The common picture here is to imagine the sheets V_α lying above U in some sense, in a sort of stack that projects down onto U

Definition.

A continuous surjection $p : E \twoheadrightarrow B$ is a *covering map* if for every $b \in B$ there exists an evenly covered neighbourhood. In this case the pair (E, p) is called a *covering space* of B.

Note.

This definition may seem a bit unintuitive – why in the space of all possible types of maps to consider should we bother with covering maps? The historical reason in my view requires a bit of complex analysis to bring out[a] but there is a good intuitive reason to care about them from the point of view of homotopy theory. The idea is that often the structure of the fundamental group can be quite thorny on our base space, with holes allowing loops to get all tangled up, so we lift everything upwards to a space where that ability to loop around something just becomes the ability to move between parts of a covering space, thus encoding the fundamental group structure in the shape of the covering space. This is a good picture to keep in mind as we go ahead

[a] For those of you that are interested: we can often understand multivalued functions as functions on a covering space. Moreover, any holomorphic map of compact Riemann Surfaces is, up to the removal of a finite set of points, a covering map

Proposition.

A product of covers is a cover, that is if (E, p) and (E', p') are covers of B and B' respectively, then $(E \times E', p \times p')$ is a cover of $B \times B'$.

Proposition.

For a connected cover (E, p), the cardinality $|p^{-1}(b)|$ is independent of $b \in B$. If it is finite we call p a finite cover, the number $k = |f^{-1}(b)|$ is called its degree, and p is a k-fold covering

Example.

Some examples of covering spaces include:[a]

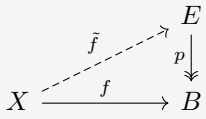
- $p : \mathbb{R} \to S^1$ via $t \mapsto e^{2\pi i t} = \cos t + i \sin t = (\cos t, \sin t)$
- $p : S^1 \to S^1$ via $z \mapsto z^n$ i.e. $(\cos t, \sin t) \mapsto (\cos nt, \sin nt)$. This is an n-fold covering and also makes clear that spaces can cover themselves
- $p : \mathbb{R}^2 \to S^1 \times S^1$ via the product of the first cover with itself. Exercise: can you see a way to relate this to the quotient space $\mathbb{R}^2 / (x, y) \sim (x + m, x + n), m, n \in \mathbb{Z}$
- $p : S^n \to \mathbb{RP}^n$ given by the quotient map (you're all familiar with this one)
- $p : \mathbb{C}^{n+1} \to \mathbb{CP}^n$ again via the quotient map
- Any homeomorphism is a 1-sheeted covering, and every 1-sheeted/1-fold covering is a homeomorphism

[a] I recommend if any of these aren't immediately obvious that you see if you can work through what the sheets are rigorously

2.2 Lifting

Definition.

Suppose we have a cover $p : E \to B$ and a continuous map $f : X \to B$. A *lift* $\tilde{f}$ of f is a map making the following diagram commute:



Note.

Lifts are why we care about covering spaces – they allow us to transfer possibly complicated maps to our complicated space to hopefully simpler ones upstairs. This works because as we'll see we have very precise criteria for when lifts exist for covering maps.[a] Lifts are generally not unique, but are unique if you fix where one point goes. The key initial theorem here is:

[a] This might seem confusing as I've only defined lifts for covering maps, but in fact a similar concept makes sense for any local homeomorphism and uniqueness still holds when existence does. Exercise: every covering map is a local homeomorphism

Theorem.

If $p : E \to B$ is a cover, $\gamma : I \to B$ is a path starting at $b = \gamma(0)$, and $e \in p^{-1}(b)$, then there exists a unique lift $\tilde{\gamma} : I \to E$ with $\tilde{\gamma}(0) = e$.

Note.

This property, at least in the case of manifolds, actually uniquely characterises covering maps!

The next fact in conjunction with this one is why lifting plays so nicely with homotopy for covering spaces:

Theorem.

Let $F : I \times I \to B$ be continuous with $F(0,0) = b$, let $p : E \to B$ be a covering map, and let $e \in p^{-1}(b)$. Then there exists a unique lift $\widetilde{F} : I \times I \to E$ such that $\widetilde{F}(0,0) = e$.

Corollary.

Homotopies lift under covers; that is, if $f \simeq g : X \to B$ and $p \circ \tilde{f} = f, p \circ \tilde{g} = g$ for a cover $p : E \to B$, then $\tilde{f} \simeq \tilde{g}$

Note.

Although paths always lift, a loop need not lift to a loop. For example, the loop going once around S^1 lifts to a path from $0 \rightsquigarrow 1$ in the cover $\mathbb{R} \to S^1$. We do however have a very precise criterion for when they do, which we'll get to. For now let's exploit this irregularity. A lift of a loop always has to start and end in the same *fibre* – preimage of a point – and so we get:

Definition.

Given a cover $p : (E, e_0) \to (B, b_0)$ we have a map[a] $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ given by $\phi([\gamma]) = \tilde{\gamma}(1)$ where $\tilde{\gamma}$ is the unique lift of γ starting at e_0, well defined by the fact that homotopies lift and in particular path homotopies lift to path homotopies which still fix endpoints.

[a] NOT a homomorphism as the codomain isn't a group

Note.

You don't need to know this for this class, but the lifting correspondence is actually a special case of the more general concept of the *Monodromy Action* – this is an action of $\pi_1(B, b_0)$ on $p^{-1}(b_0)$ given by $[\gamma](e_i) = \tilde{\gamma}(1)$ where $e_i \in p^{-1}(b)$ and $\tilde{\gamma}$ is the unique lift of γ starting at e_i. This is very useful but not relevant to this class

There are a couple propositions about the lifting correspondence ϕ that reflect the fact that it encodes a lot of useful information about the cover and the fundamental group. There are:

Proposition.

- If E is path-connected then $\phi : \pi_1(B, b_0) \twoheadrightarrow p^{-1}(b_0)$ is surjective.
- If E is simply-connected then $\phi : \pi_1(B, b_0) \xrightarrow{\sim} p^{-1}(b_0)$ is bijective.

Exercise.

If you didn't do it this way on the problem set, show that any path connected cover of a simply connected space is a homeomorphism using the first of the above propositions.

We now have the tools to determine $\pi_1(S^1)$:

Theorem.

$\pi_1(S^1) \cong \mathbb{Z}$

Proof.

Consider the covering map

$$p : (\mathbb{R}, 0) \longrightarrow (S^1, (1,0)), \qquad p(x) = (\cos 2\pi x,\ \sin 2\pi x).$$

Since $\mathbb{R}$ is simply connected, by the above proposition the map

$$\phi : \pi_1\big(S^1, (1,0)\big) \longrightarrow p^{-1}\big((1,0)\big) = \mathbb{Z}$$

is a bijection. It remains to show that ϕ is a group homomorphism.

Let $[f], [g] \in \pi_1(S^1, (1,0))$ and write $\phi([f]) = n$ and $\phi([g]) = m$. That is, the lifts $\widetilde{f}$ and $\widetilde{g}$ starting at 0 end at n and m, respectively.

Define a new path $h : I \to \mathbb{R}$ by

$$h(s) = n + \widetilde{g}(s).$$

Then h is the lift of g starting at $n = \widetilde{f}(1)$. Hence the concatenation $\widetilde{f} * h$ is a well-defined path in $\mathbb{R}$ from 0 to $n + m$, and it is the lift of $f * g$ starting at 0. Therefore,

$$\phi([f * g]) = n + m = \phi([f]) + \phi([g]).$$

Thus ϕ is a group homomorphism. ■

With that cool fact out the way, we return to the general theory. We want to understand when loops lift to loops and when maps in general lift. The second one in particular is more useful than you could ever imagine, I promise. The key fact and associated object for both of these is the following:

Theorem.

For a covering $p : E \to B$, $p_* : \pi_1(E) \to \pi_1(B)$ is injective.

Proof.

Any nulhomotopy of a loop of the form $p \circ \tilde{h}$ lifts to a nulhomotopy of $\tilde{h}$ and thus $\ker p_*$ is trivial. ■

Thus we have associated to any cover $p : E \to B$ a subgroup $p_*(\pi_1(E))$ isomorphic to $\pi_1(E)$; it will turn out that this uniquely identifies the covering space up to a suitable notion of equivalence. This allows us to phrase the relevant theorems:

Theorem. Lifting Lemma for Loops

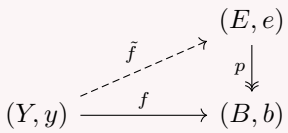
For a cover $p : E \to B$, a loop $\gamma : I \to B$ lifts to a loop in $\pi_1(E)$ if and only if $[\gamma] \in p_*(\pi_1(E))$.

Note.

The next theorem is for me tied with the First Isomorphism Theorem for the theorem I spam cite the most. I really cannot overemphasise how many times you will need this theorem and thus it is important to remember which direction things go here:

Theorem. Lifting Criterion

Suppose we have a cover $p : (E, e) \to (B, b)$ and a continuous map $f : (Y, y) \to (B, b)$. Then a lift $\tilde{f} : (Y, y) \to (E, e)$ making the following diagram commute:



exists if and only if $f_*(\pi_1(Y)) \subset p_*(\pi_1(E))$ as subgroups of $\pi_1(B)$. If it exists, it is unique

Note.

Uniqueness only holds for *pointed* topological spaces; that is, the lift is unique once we choose a particular preimage of $b = f(y)$ as our basepoint for E. Moreover, it is very important to remember which direction the inclusion goes. I like to think about it like this: the portion of $\pi_1(B)$ that plays nicely with p is exactly $p_*(\pi_1(E))$, and so if we want f to lift under p the image of its induced map isn't allowed to leave $p_*(\pi_1(E))$. Thus $f_*(\pi_1(Y)) \subset p_*(\pi_1(E))$.

2.3 Classification of Coverings

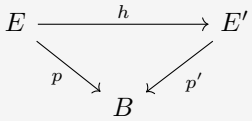
To further hammer in the significance of $p_*(\pi_1(E))$, we can show that p is uniquely determined by it. We need a good notion of equivalence for this; the obvious one is:

Definition.

Let $p : E \to B$ and $p' : E' \to B$ be coverings. We say that p and p' are *equivalent* if there exists a homeomorphism $h : E \to E'$ such that

$$p = p' \circ h.$$

In this case we call h an *equivalence of coverings*. Equivalently, the diagram



commutes.

Note.

If we define a morphism of covering spaces of B $(E,p) \to (E',p')$ to be a map $E \to E'$ making the above diagram commute, the coverings of B form a (small) category. Equivalences of coverings are exactly isomorphisms in this category

Note.

From here on out we assume everything to be path-connected and B to always be locally path connected. You don't actually have to worry too much about these technical conditions, but see the lecture notes or ask me if you want more on them

Theorem.

Let $p : E \to B$ and $p' : E' \to B$ be covering maps with $p(e_0) = p'(e_0') = b_0$. There exists an equivalence of coverings $h : E \to E'$ such that $h(e_0) = e_0'$ if and only if the subgroups

$$H = p_*\big(\pi_1(E, e_0)\big) \qquad \text{and} \qquad H' = p'_*\big(\pi_1(E', e_0')\big)$$

are equal (i.e. the same subgroup of $\pi_1(B, b_0)$). Moreover, if such an h exists, then it is unique.

Note.

The induced subgroups of $\pi_1(B)$ have to be *the same,* not just isomorphic.

This says that, fixing base points, there are unique equivalences between coverings if and only if their induced subgroups are the same. We can say more: if we relax either condition a bit, we get:

Theorem.

Let $p : E \to B$ and $p' : E' \to B$ be covering maps with $p(e_0) = p'(e_0') = b_0$. Then p and p' are equivalent if and only if the subgroups

$$H = p_*\big(\pi_1(E, e_0)\big) \qquad \text{and} \qquad H' = p'_*\big(\pi_1(E', e_0')\big)$$

are conjugate subgroups of $\pi_1(B, b_0)$.

Note.

It's a bit beyond the scope of this course, but in fact we can (assuming B is semilocally simply connected, which the Hawaiian earring space we talked about for example isn't) for any subgroup of $\pi_1(B)$ construct a corresponding covering space with that induced subgroup; thus there is for such a B a one-to-one correspondence between the set of covers of B up to equivalence and the set of conjugacy classes of subgroups of B. In fact with what we're about to see we'll see that if we instead make this a one-to-one correspondence of based covers with subgroups and consider the category of such covers and the poset category of subgroups with inclusion, this correspondence is an equivalence of categories! If this doesn't make sense to you, don't worry, it's not important.

Now, a special instance of this correspondence is very important: the case $p_*(\pi_1(E)) = \{1\}$, that is, simply connected covers.

2.4 Universal Covers

Definition.

The *Universal Cover* of a topological space B is the, unique up to equivalence if it exists, simply-connected covering space of B

Proposition.

If B is semilocally-simply-connected,[a] the universal covering space always exists.[b]

[a] a technical condition very few people spend time on. If you're interested, the condition is that the topological space has a basis consisting of simply connected opens

[b] via an explicit construction very few people spend time on

Example.

An example of a nonlocally simply connected space is the "Hawaiian Earrings" space, given by:

$$\mathbb{HE} := \bigcup_{n \in \mathbb{Z}^+} C_n$$

Where C_n is a circle of radius $\frac{1}{n}$ centered at $\frac{1}{n}$. You don't really need to worry about such spaces.

Example.

Here are some examples of universal covers:

- $p : \mathbb{R} \to S^1$ as above
- $p \times p : \mathbb{R}^2 \to S^1 \times S^1$
- The universal covering space of the figure eight space $S^1 \vee S^1$:

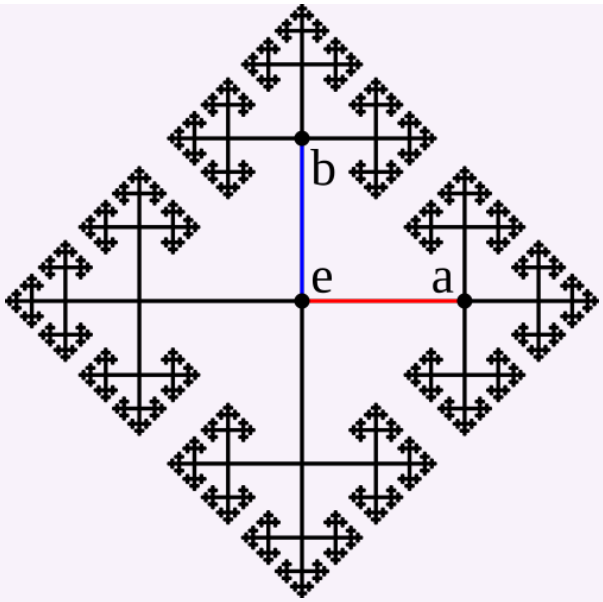


Figure 1: The universal cover of $S^1 \vee S^1$, which coincides with the Cayley graph of $\pi_1(S^1 \vee S^1) \cong F_2$. This isn't always the case, but isn't a coincidence either!

> This one bears some explanation. The idea is that if we label the two loops on the figure 8 a and b, then the horizontal edges all map to a and the vertical ones to b; every nontrivial loop lifts to a path corresponding to a sequence of steps on this graph, and the nodes are exactly the fiber of the singular point in the middle of $S^1 \vee S^1$. This one makes you see very visually how often the universal cover captures the behaviour of the fundamental group in its geometry.

The sense in which these covers are universal is as follows:

Theorem.

Let $p_0 : E_0 \to B$ be a universal covering, and let $p' : E' \to B$ be any path-connected covering. Then there exists a covering map $q_0 : E_0 \to E'$ such that

$$p' \circ q_0 = p_0,$$

and moreover q_0 is a universal covering of E'.

Proof.

Consider the lifting problem

$$\begin{array}{ccc} & & (E', e') \\ & \overset{q_0}{\nearrow} & \big\downarrow{\scriptstyle p'} \\ (E_0, e_0) & \xrightarrow{p_0} & (B, b) \end{array}$$

Since all spaces are simply connected, by the Lifting Criterion the lift q_0 automatically exists. It then suffices to show that it is a covering. There's a full proof of this in Munkres, Lemma 80.2 that I'll direct you to. For the case where everything is a manifold however, which is many of the spaces we care about, I can modulo a certain lemma give you a proof that should give you an idea of why this is true,

and that uses a fun fact I'd like to share in any case. In particular, I referenced the fact above that any local homeomorphism of manifolds with the property that any path lifts is in fact a covering map. If you want a relatively elementary proof of this, I'll refer you to page 28 of Forster's *Lectures on Riemann Surfaces*, it's Theorem 4.19. In any case, if you're willing to accept this, we can prove this fact. q_0 is clearly a local homeomorphism, as the composition of a local homeomorphism with something that isn't is never a local homeomorphism. So it suffices to show any curve lifts under q_0. Take a curve $\gamma : (I, 0) \to (E', e')$ – without loss of generality assuming that we chose our basepoint to be the starting point of the curve. Then the path $\alpha = p' \circ \gamma : (I, 0) \to (B, b)$ must lift to a path $\tilde{\alpha} : (I, 0) \to (E_0, e_0)$ with the property that $p_0 \circ \tilde{\alpha} = p' \circ \gamma$. Then by commutativity of the above diagram:

$$p' \circ q_0 \circ \tilde{\alpha} = p_0 \circ \tilde{\alpha}$$
$$p' \circ q_0 \circ \tilde{\alpha} = p' \circ \gamma$$

So $q_0 \circ \tilde{\alpha}$ and γ are both lifts of $p' \circ \gamma$ starting at the same point and as such by uniqueness of lifts under the cover p', $q_0 \circ \tilde{\alpha} = \gamma$ and $\tilde{\alpha}$ is a lift of γ. ■

3 Prelude to SvK

The climax of our Algebraic Topology Unit is the Seifert-van-Kampen Theorem, which we'll get to next week. We did see some hints of it this week though, in particular the following special case of it:

Proposition.

Suppose $X = U \cup V$, with U and V open, $U \cap V$ path-connected, and $x_0 \in U \cap V$. Let $i : U \hookrightarrow X$ and $j : V \hookrightarrow X$ be the inclusion maps. Then the images of

$$i_* : \pi_1(U, x_0) \to \pi_1(X, x_0) \qquad \text{and} \qquad j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$$

generate $\pi_1(X, x_0)$.

This reflects that we can with enough effort break the study of the fundamental group of a space down into the study of the fundamental groups of open subsets of it, up to some intervening term that we have to quotient by. We'll elaborate this more next week, but for now consider the corollary:

Corollary.

If $X = U \cup V$ is path-connected, $\pi_1 U \cong \pi_1 V \cong \{1\}$, and $U \cap V$ is path connected,[a] then X is simply connected.

[a] one could say if you really wanted to spruce this up $|\pi_0(U \cap V)| = 1$

Corollary.

$\pi_1(S^n) = \{1\}$ for all $n \geq 2$

Proof.

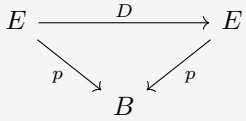
Divide S^n into hemispheres. ■

4 Deck Transformations (NOT COURSE CONTENT)

What follows is, and I cannot make this clear enough, ***not part of the content of this course.*** You do not need to know this for any problem sets, you will not be assessed on it, and Auroux does not expect you to know it. You also should not panic if some of what follows does not make sense. It also may well be confusing at first. I include it only because it is adjacent to the content covered in this review session and because in future classes I took it ended up being supremely inconvenient that I did not know what this was, and the theory is quite beautiful. It's a couple definitions and nice results, and they're good to know especially if you ever want to do anything with Riemann Surfaces. In any case, let's make the standard assumption that everything involved is path-connected and get going:

Definition.

A *deck transformation* of a covering $p : E \to B$ is an automorphism[a] $D : E \xrightarrow{\sim} E$ making the following diagram commute:



That is, D is an equivalence of the covering with itself, with no requirement that basepoints are fixed.[b] Equivalently, a Deck Transformation is an automorphism of (E, p) in the category of coverings of B described earlier.

[a] homeomorphism of the space to itself
[b] In fact we will see that a deck transformation with a fixed point must be identity

Definition.

The *deck group*, which is denoted by any of $\mathrm{Aut}(E/B), \mathrm{Gal}(E/B)$, or $\mathrm{Deck}(E/B)$, is the group of deck transformations under composition. This is not standard but for clarity if the cover is not clear just from the spaces involved I will use a subscript p to indicate that its with respect to a particular map

Note. For you 123ers

The E/B notation here is ***not*** a quotient. Rather it represents the idea of E being "over" B, just like we say V/k for a vector space V over a field k or, for those of you that have taken 123, L/K for a field extension L of K. This is, in the second case, more than just a coincidence! In fact there are deep analogies between coverings and field extensions, and between the galois group of a field extension and the deck group of a covering. In certain cases[a] these analogies even amount to an equivalence of categories, but that is a very deep theorem in the theory of Riemann Surfaces. If you're interested, I cannot recommend MATH213BR enough! Especially if Curtis McMullen is teaching it.

[a] Branched covers of $\widehat{\mathbb{C}}$, although there is no reason for you to know what those are yet

Proposition.

A deck transformation is determined by its image on a single point

Proof.

Let this single point be $e \in E$, and take any $e' \in E$. Then take a path $\gamma : e \rightsquigarrow e'$. $D \circ \gamma$ is a path $D(e) \rightsquigarrow D(e')$ with $p \circ D \circ \gamma = p \circ \gamma$. Thus $D(e')$ can be recovered from $D(e)$ as the endpoint of the

unique lift of the path $p \circ \gamma$ starting at $D(e)$. ■

Corollary.

No nonidentity Deck Transformation has a fixed point

Note.

With this in mind, this gives a nice way of thinking about deck transformations: they are permutations of the fibres! That is, they are automorphisms such that their restriction to any fibre $p^{-1}(b), b \in B$ is a permutation of $p^{-1}(b)$. Since $D \circ p = D$ any deck transformation must do this, and by the above proposition deck transformations must be determined by this permutation

Example.

Some examples:

- $\text{Deck}(\mathbb{R}/S^1) \cong \mathbb{Z}$
- $\text{Deck}(\mathbb{R}^2/S^1 \times S^1) \cong \mathbb{Z}^2$
- $\text{Deck}(C/S^1 \vee S^1) \cong F_2$, where C is the universal cover
- For the cover of S^1 by itself given by $z \mapsto z^n$, $\text{Deck}_p(S^1/S^1) \cong {}^{\mathbb{Z}}\!/_{n\mathbb{Z}}$ and is the multiplicative group generated by multiplication by the principle nth root of unity ω

You might notice that in the cases where these are universal cover that the Automorphism group coincides with the deck group. This is not accidental!

There is a particular subclass of coverings whose Deck Groups are particularly well behaved. The name is very suggestive:

Definition.

A cover $p : E \to B$ is called *normal, regular,* or *Galois* if the action of $\text{Deck}(E/B)$ on fibres is transitive. That is, for any $b \in B$ and $e, e' \in p^{-1}(b)$, there exists $D \in \text{Deck}(E/B)$ with $D(e)$.

Note.

If such a D exists for some e, e', it is unique by the above proposition that D is uniquely specified by its action on a point.

It is a fact that we'll prove in a moment that universal covers are always normal. This is important because of the following wonderful result:

Theorem.

Given a normal covering $p : E \to B$, we have that:

$$B \cong {}^{E}\!/_{\text{Deck}(E/B)} := {}^{E}\!/_{(e \sim D(e) \forall D \in \text{Deck}(E/B))}$$

This is called the quotient of E by the action of the Deck group

Proof.

For a normal covering, two elements of E are in the same fibre of p exactly when there exists a Deck Transformation mapping one to the other, and this together with the fact that p is a local homeomorphism means that $p : E \to B$ is a quotient map for this equivalence relation. ■

Thus we care a lot about which coverings are normal, which the next result will nicely classify. The following result requires you to remember the definition of the *normaliser* in group theory. To remind you:

Definition.

Given a subgroup H of a group G, its *normaliser* $\mathcal{N}(H) \leq G$ is the largest subgroup[a] of which H is a normal subgroup. Explicitly:

$$\mathcal{N}(H) := \{g \in G \mid gHg^{-1} = H\}$$

If $H \trianglelefteq G$ is in fact normal, then $\mathcal{N}(H) = G$.

[a] in the sense of inclusion; any other subgroup with this property is a subgroup of $\mathcal{N}(H)$

We are now ready to state our main results classifying which covers are Galois:

Proposition.

Given $e_0, e_1 \in p^{-1}(b)$ for some b, there exists $D \in \mathrm{Deck}(E/B)$ with $D(e_0) = e_1$ if and only if the homotopy class of any loop $\gamma = p \circ \alpha$ where α is a path $e_0 \rightsquigarrow e_1$ is in $\mathcal{N}(p_*\pi_1(E, e_0))$

Proof.

Observe that D is a lift of $p : (E, e_0) \to (B, b)$ through $p : (E, e_1) \to (B, b)$, and as such by Lifting Criterion exists if and only if $p_*\pi_1(E, e_0) \subset p_*\pi_1(E, e_1)$. But notice that these subgroups are conjugate to each other by any $\gamma \in \pi_1(B, b)$ that lifts to a path $e_0 \rightsquigarrow e_1$, and as such for one to be a subset of the other any such γ must conjugate $p_*\pi_1(E, e_0)$ to a subset of itself[a] and thus $\gamma \in \mathcal{N}(p_*\pi_1(E, e_0))$. ■

[a] conjugation is an isomorphism

Corollary.

$p : E \to B$ is a normal covering if and only if $p_*\pi_1(E) \trianglelefteq \pi_1(B)$ is a *normal* subgroup of $\pi_1(B)$.

Proof.

For p to be normal $\mathrm{Deck}(E/B)$ must act transitively on fibres, so by the previous proposition since any $\gamma \in \pi_1(B, b)$ lifts to *some* path between fibres every $\gamma \in \pi_1(B, b)$ must in fact satisfy $\gamma \in \mathcal{N}(p_*\pi_1(E, e_0))$ and conversely this is sufficient. Thus $\mathcal{N}(p_*\pi_1(E, e_0)) = \pi_1(B, b)$ and $(p_*\pi_1(E, e_0)) \trianglelefteq \pi_1(B, b)$ is a normal subgroup. ■

Corollary.

Universal covers are always normal

Proof.

The trivial group is always a normal subgroup. ■

Proposition.

For a Normal covering:

$$\text{Deck}(E/B) \cong {}^{\pi_1(B)}\!/_{p_*\pi_1(E)}$$

Proof.

This follows almost directly from the correspondence used in the above proof between loops in B, paths between points of a fiber and, elements of $\text{Deck}(E/B)$. We have a map $\rho : \pi_1(B) \twoheadrightarrow \text{Deck}(E/B)$ given by the Monodromy action – that is, given $\gamma \in \pi_1(B)$ we can specify $\rho(\gamma)(e_0) = \tilde{\gamma}_{e_0}(1)$ as usual, and this will define permutations of every fibre as desired. The kernel of this map is exactly the loops that lift to loops and thus whose associated deck transformation has a fixed point and is thus identity. By path-lifting lemma these are exactly the members of $p_*\pi_1(E)$. The first isomorphism theorem then gives our result. ■

This all combines to give our *coup de grâce:*

Theorem.

Every space satisfying certain weak regularity conditions[a] is the quotient of its universal cover by the action of the Deck Group, which is isomorphic to its fundamental group

[a] locally-simply connected path-connected locally path-connected – I'm just putting these all here to cover my technical bases, they're not all that important

Proof.

Any such space B has a universal cover E, which as above is automatically a normal cover and as such the space is isomorphic to the quotient of the universal cover by the Deck group. Moreover by the other proposition above:

$$\text{Deck}(E/B) \cong {}^{\pi_1(B)}\!/_{p_*\pi_1(E)} \cong {}^{\pi_1(B)}\!/_{\{1\}} \cong \pi_1(B)$$

■

Corollary.

- $S^1 \cong {}^{\mathbb{R}}\!/_{\mathbb{Z}}$
- $S^1 \times S^1 \cong {}^{\mathbb{R}^2}\!/_{\mathbb{Z}^2}$
- $S^1 \vee S^1 \cong {}^{C}\!/_{F_2}$ where C is the Cayley graph of F_2 and F_2 acts in the natural way
- $\mathbb{RP}^n \cong {}^{S^n}\!/_{x \sim -x}$

There is however one more thing you can say. You'll recall that Auroux said that we can construct a covering space with $p_*\pi_1(E, e) = H$ for any subgroup $H \leq \pi_1(B, b)$. We're now ready to prove this! Again I am assuming all the usual niceness properties

Theorem.

For any subgroup $H \leq \pi_1(B, b)$, there exists a covering space $p : (E, e) \twoheadrightarrow (B, b)$ with $p_*\pi_1(E, e) = H$. With earlier results, this gives an equivalence of categories between the category of based covers of (B, b) and the poset category of subgroups of $\pi_1(B, b)$.

Proof.

Let $q : (\tilde{B}, \tilde{b}) \twoheadrightarrow (B, b)$ be the universal cover and $\mathrm{Deck}(\tilde{B}/B) \cong \pi_1(B, b)$ the deck group. Then we can realise $H \leq \pi_1(B, b)$ as a subgroup $H \leq \mathrm{Deck}(\tilde{B}/B)$, and take the quotient $E = \tilde{B}/H$ to retrieve a new covering space $p = \bar{q} : (E, [\tilde{b}]) \twoheadrightarrow (B, b)$. The subgroup $p_*\pi_1(E, [\tilde{b}])$ is exactly the subgroup of $\mathrm{Deck}(\tilde{B}/B)$ that acts trivially on the quotient space $\tilde{B}/H$ by the lifting lemma, which by construction is H. ■