

55B Week 9 Review Notes

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1 The Implicit and Inverse Function Theorems

The two theorems covered in this section are core to what makes smooth functions so nice. In particular, many of their nice topological properties come from the fact that wherever they have nonzero derivative they are local diffeomorphisms, and much of the reasoning we do about graphs of curves as topological spaces – in particular the reason that they are often manifolds especially in the case of polynomials – comes from the implicit function theorem. First, some preliminaries:

Theorem. Chain Rule

If g is differentiable at $x \in \mathbb{R}^n$, and f is differentiable at $g(x) \in \mathbb{R}^m$, then $f \circ g$ is differentiable at x and:

$$D(f \circ g)(x) = (Df)(g(x)) \circ (Dg)(x)$$

Note.

If we generalise to the case of differentiable manifolds while specialising to the case of everywhere smooth maps $f : M \rightarrow N$, this expresses the fact that the assignment of a differentiable manifold M to its tangent bundle $TM = \coprod_{p \in M} T_p M$ and of smooth maps to their total derivative $df : TM \rightarrow TN$ is a functor.

One might hope that other nice properties of one variable differentiable maps, in particular the mean value theorem, would continue to hold. Unfortunately higher dimensions are flexible enough that the full MVT fails, however we can retain:

Theorem. Mean Value Inequality

If $f : U \rightarrow \mathbb{R}^m$ is differentiable at every point of the line segment:

$$[a, b] := \{(1-t)a + tb : t \in [0, 1]\}$$

Then we have:

$$|f(b) - f(a)| \leq |b - a| \sup_{x \in [a, b]} \|Df(x)\|$$

Note.

Be careful! $\|Df(x)\|$ here is the *operator norm*, not any kind of Euclidean norm.

Now a key fact that is supremely convenient for calculations and will also be extremely important for a fundamental result about differential forms later:

Proposition.

If for all i, j the partials $\frac{\partial^2 f}{\partial x_i \partial x_j}$ exist and are continuous then:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

Note.

One way of saying this is the *Hessian Matrix* whose entries are these double partials is symmetric; it can be thought of as representing a symmetric bilinear form on tangent vectors

We now move to our main theorems. Recall the idea that $Df(x)$ is the “best linear approximation to f near x ”. In particular, since this approximation gets arbitrarily good close to x , the local behaviour of f is entirely determined by Df , and so if $Df(x)$ is injective then on a sufficiently small neighbourhood of x and similarly if $Df(x)$ is surjective then f maps a neighbourhood of x surjectively onto a neighbourhood of $f(x)$ (this is not trivial! it could be that neighbourhoods near x are collapsed into nonopen spaces when $Df(x) = 0$, or that they map into a space where they are no longer open). Combining these will give a powerful theorem called the inverse function theorem. For this, we need the right language, namely:

Definition.

A map $f : U \rightarrow V$ between open subsets of $\mathbb{R}^n$ is^a a diffeomorphism if it is a C^1 bijection with C^1 inverse.

^amore generally a map between open subsets of any two equal-dimensional manifolds

This gives us the language for an immensely powerful theorem that will be of great use in complex analysis:

Theorem. Inverse Function Theorem

Let $p \in E \subset \mathbb{R}^n$ with E open, $f \in C^1(E, \mathbb{R}^n)$, and suppose $Df(p) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isomorphism, equivalently $\det Df(p) \neq 0$. Then f is a local diffeomorphism at p , that is there exists a neighbourhood $U \ni p$ such that $f|_U : U \xrightarrow{\sim} f(U)$ is a diffeomorphism.

I won't go over the whole proof again, but will note that the core idea is an application of the contraction mapping principle paired with the mean value inequality, which is a taste of how ubiquitous they are in multivariate analysis. What this theorem expresses precisely is exactly that idea that the local behaviour of a function, which is characterised by whether or not it's a local homeomorphism and the degree to which that fails, is fully specified by its derivative, and that functions with nonzero derivative are locally very well behaved. An immediate consequence of this theorem that is kind of hard to state but not hard to prove and is very important in course after this one is:

Theorem. Implicit Function Theorem

Suppose we have an open subset $U \subset \mathbb{R}^n \times \mathbb{R}^m$ and a differentiable map $f : E \rightarrow \mathbb{R}^m$, where we write for $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ that $(x, y) \mapsto f(x, y)$. We have a direct sum decomposition $Df(x, y) = Df_x \oplus Df_y : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$. Suppose moreover that $f(x_0, y_0) = 0$ and Df_y is invertible – as usual this is equivalent to having nonzero determinant – at $(x_0, y_0) \in E$. Then there exist neighbourhoods U, V of x, y in $\mathbb{R}^n \cap E, \mathbb{R}^m \cap E$ respectively such that for all $x \in U$ there is a unique y with $f(x, y) = 0$, meaning we may write it as a function of x , $y = g(x)$, $f \circ (\mathbb{1} \times g) \equiv 0$. Moreover, if we define $g : U \rightarrow V$ in this way, g is differentiable on U and:

$$Dg = -(Df_y)^{-1} Df_x$$

Note.

This theorem can be a little bit difficult to digest at first because there are so many symbols being thrown at you. Fundamentally what the theorem is saying is a very simple statement: If a curve is defined as the zero set of some multivariate differentiable function, and the last few coordinates of this function make up a function with nonzero derivative, then locally the curve is just the graph of a smooth function, and we know what that smooth function looks like. A good test of whether you've digested this and one or two other definitions is to check whether you can prove the following statement:

Exercise.

Prove that the zero-set $V(f)$ of a differentiable function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $(x, y) \mapsto f(x, y)$ such that $Df(x, y) \neq 0$ for all $(x, y) \in V(f)$ is a manifold. If you know what this is, prove that it's a smooth manifold

2 Integration in several variables

This section is really just a prelude to the section on differential forms, since those are really what this is all about. The key point is that it is at first very easy to define integrals of functions on n -cells – products of intervals $I = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$. Namely, you can either split it into small cubes and do the usual Riemann approximation, which works so long as the volume of the region on which f is not continuous goes to 0 for finer and finer subdivisions, or just by iterated integration. These coincide, and Frubini's theorem

tells us that iterated integrals do not depend on the order of integration. However, we run into some issues for how to define integrals over more exotic hypersurfaces.¹ Of course, we could just define f to be zero outside the region and thus extend it to an n -cell, but this is a bit unnatural. A better way is by change of coordinates. The way to phrase this using our current language is:

Theorem. Change of Variables

If $\varphi : U \rightarrow V$ is a diffeomorphism, f continuous on V , then, indicating by $|dy| = dy_1 \cdots dy_n$ the appropriate dimensional collection of coordinates:

$$\int_V f(y)|dy| = \int_U f(\varphi(x))|\det D\varphi(x)||dx|$$

Note.

The intuition for this formula is that $|\det D\varphi(x)|$ is the scaling factor by which small cuboids will scale. However, there's a lot not to like about how we've phrased this formula and defined integration in several variables so far. There are a lot of random quantities that pop up that aren't particularly well motivated, and moreover, if we want to extend this theory to general manifolds we'll run into the problem that as we've defined it integration is a fundamentally coordinate-dependent endeavour – consider for example integrating the constant function over a given region C ; this just returns the volume of that region, which is highly dependent on the choice of coordinates. It is at this point that we'll find out that the idea that dx is just a symbol is a *lie*, and getting past this convenient falsehood makes the theory of multivariate integration a lot cleaner. For example, for those of you who know some multivar, what differential forms allow us to do is phrase a grand theorem that in one line simultaneously proves Stokes' theorem, Green's theorem, the fundamental theorem of line integrals, and generalises them to every dimension. It will however involve some formal symbol pushing though, so buckle up.

3 Differential Forms

3.1 The Definition

Before we get into the formalism of what a differential form is, I'd like to give a little bit of an intuitive explanation of what's going on with them. Roughly, differential forms are meant to be a formalism for the kind of thing we can integrate. What kind of properties should this have? Fundamentally, the idea of integration is to sum up lots of infinitesimal volumes (one could also think about this as densities at these points) together along a certain oriented surface and get the result. Thus what we need an object to satisfy for it to qualify as something we could integrate is that it assigns to every point a way of measuring an infinitesimal volume element at that point. There are two words we need to make precise here. The first is infinitesimal. As you have all probably (rightly!) heard at some point, infinitesimals are scary and worth avoiding if possible. What we *really* mean when we use the language of infinitesimals is the behaviour of an object to first order, where we assume everything is linear; that is, we want to consider things on the *tangent space*, much like how $Df(x)$ is a function on the tangent space.² So we want to be able to measure volume on the tangent space. But the tangent space³ is a vector space, and we know how to measure k -dimensional

¹regions parametrised via smooth functions by n -cells

²*exactly* like that, as we'll come to see

³however you define it; there's actually a lot of subtlety here. To be entirely honest I don't fully understand the equivalences between the various definitions of the tangent space myself. The most intuitive one for general manifolds is in my view as

volume in a vector space! In particular, a k -volume form is a function that assigns to a collection of k vectors in some vector space V a volume in a k -multilinear way that vanishes whenever the vectors have a linear relation⁴; that is, a k -volume form is just an element of $\bigwedge^k V^*$. In our case our vector space is just the tangent space T_p – which in our case will always just be $\mathbb{R}^n$ – and so what we want is an assignment, for every point p we care about, of that point to a particular member of $\bigwedge^k T_p^*$. We're now ready for our definition!

Definition.

Given an open subset^a $U \subset \mathbb{R}^n$, a *differential k -form* ω on U is a map:

$$\omega : U \rightarrow \prod_{p \in U} \bigwedge^k T_p^*$$

With the property that $\omega(p) \in T_p^*$ for all $p \in U$. We say that ω is a *smooth k -form* if this map is C^∞ with respect to a certain smooth manifold structure on $\prod_{p \in U} \bigwedge^k T_p^*$; we'll get a more concrete definition of this in a second so don't worry too much about it. We define C^0, C^1 forms similarly.

^aIn general of a manifold M

Note.

The condition that $\omega(p) \in T_p^*$ for all $p \in U$ can be phrased in a more abstract way. We call the union $\coprod_{p \in U} T_p^*$ the *cotangent bundle* over our space. You'll hear the word bundle a lot in math – it's just a word for when to every point of a topological space we associate a similar structure – and a bundle naturally has a projection map π associated to it that sends any element in this big union to the unique point of the topological space above which it lives. That projection map is most definitely a surjection, and the condition $\omega(p) \in T_p^*$ for all $p \in U$ is just saying that ω is a right inverse to π . For historical reasons, we call a right inverse a *section*. As such, we may call a differential 1 form a *section of the cotangent bundle*, and a k -form is a section of the k 'th exterior power of the cotangent bundle.

Now observe that given an open subset $U \subset \mathbb{R}^n$ the differentials/total derivatives $dx_i(p) = Dx_i(p)$ of the coordinate maps at any point $p \in U$ are elements of T_p^* (we'll get back to this concept, it's very important), and in fact form a basis of the vector space, being the standard dual basis to the standard basis of the tangent space. Thus any element of $\bigwedge^k T_p^*$ can be written as some linear combination of pure alternating k -tensors $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. Therefore, if we write an ordered multi-index $i_1 < \cdots < i_k$ – we may assume this ordering because we may permute the dx_i to get them in increasing order and just possibly pick up a negative sign – as I with $|I| = k$, and indicate by dx_I the tensor $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, we have that we may represent any k -form ω as:

$$\omega(p) = \sum_{|I|=k} f_I(x) dx_I = \sum_{i_1 < i_2 < \cdots < i_k} f_{i_1 \dots i_k}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

Proposition.

A form is C^k if and only if the functions f_I are C^k , and C^∞ if the functions f_I are C^∞

equivalence classes of curves $\gamma : (\mathbb{R}, 0) \rightarrow (M, p)$ under the equivalence relation that their derivative at 0 agrees in any chart; you can think of this as velocities at the point p . You don't really need to worry about the tangent space for general manifolds though, all you need to know is that the tangent space is a vector space and that intuitively it behaves like what we imagine when we draw tangent lines and tangent planes

⁴because if the vectors are linearly dependent then the parallelepiped is degenerate

Definition.

The space of C^∞ k -forms on U is denoted by $\Omega^k(U)$; explicitly:

$$\Omega^k(U) := \{\omega \in C^\infty(U, \wedge^k T^*) : \omega(p) \in \wedge^k T_p^*\}$$

Where the absence of the subscript p on the T^* indicates we're considering the whole bundle of cotangent spaces at every point.

Note.

In this class we use $\Omega^k(U)$ for the space of smooth k -forms, but in many other contexts Ω is reserved for what's called holomorphic 1-forms – you'll be able to guess what that is in a week or two – and we use instead the notation $\mathcal{E}^{(k)}(U)$ for the space of smooth k -forms.

Some examples quickly, just to give you an idea of what these look like:

Example.

I'm just putting these here because I think the above formula looks scary and I want you to have a couple concrete examples.

1. 1-forms

(a) $dx + dy$

(b) $\frac{x}{x^2+y^2}dx - \frac{y}{x^2+y^2}dy$

(c) $zdx + xdy + ydz$

2. 2-forms

(a) $dx \wedge dy$

(b) $xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$

(c) $x^2dx \wedge dy - \sin(y)xy^3dz \wedge dy$

(d) $4x_3dx_4 \wedge dx_1 - e^{x_1x_2}dx_2 \wedge dx_3$

3. 3-forms

(a) $4x^3ydx \wedge dy \wedge dz$

(b) $\sin(x_4)dx_1 \wedge dx_2 \wedge dx_4 + x_2dx_1 \wedge dx_3 \wedge dx_4$

3.2 Integration of Forms

Now you might ask, we defined these things so that we could integrate them, how do we integrate them! We'll get there in a second, but first you need to know one fundamental construction for differential forms that makes that definition possible, namely the *pullback* of a differential form. This is the analogue of change of coordinates for forms. If we think of forms as being the differential analogue of members of the dual vector space – which is really what they are on each tangent space – this is the analogue of the transpose linear map to a linear map, and indeed we'll use the same notation for it. The intuition you should have in the back of your mind though is that for a smooth map φ the pullback φ^* is just change of coordinates through φ , and formally just setting $x = \varphi$ and bashing computations often just works, which is part of the point of the differential forms construction. The definition is:

Definition. Pullback of Differential Forms

Given a smooth map $\varphi : U \rightarrow V$, the *pullback map* $\varphi^* : \Omega^k(V) \rightarrow \Omega^k(U)$ (note the contravariance) is for each k defined on a smooth form $\omega \in \Omega^k(V)$ by, remembering that ω is a map to an exterior power of a dual space and thus we have to specify at each x what the map does on tangent vectors:

$$(\varphi^*\omega)(x)(v_1, \dots, v_k) := \omega(\varphi(x))(D\varphi(x)(v_1), \dots, D\varphi(x)(v_k))$$

Note.

This formula can look a little scary, but I want you to notice that at each step of the definition that if we want to define a notion of “precomposing ω by φ ” we do what we *have to* – that’s where the $D\varphi(x)$ terms come from, as this is the embodiment of φ at the level of tangent spaces and thus the thing we need to be precomposing by on tangent vectors.

Proposition.

- For a 0-form, that is a function to $\mathbb{R}$, $\varphi^*f = f \circ \varphi$
- $\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta$

There’s another important property listed in the lecture notes, but I’m deferring it because I want to define things in a slightly different order to Auroux

This now allows us to define integration of forms. Once again this definition is in three steps: first, we define integration on a k -cell in the k coordinates appearing in a given pure tensor involved in the form – if the k -cell doesn’t line up with the parallelepiped specified by the form the integral is 0, so integrating $dx \wedge dy$ along the xz plane yields 0 for example – in the obvious way by iterated integration. One then defines integration of a general differential form, which is a sum of these, by extending by linearity. Then for general k -submanifold $M \subset U \subset \mathbb{R}^n$ parametrised by a smooth map from a k -cell $\varphi : D \hookrightarrow U$ with $M = \varphi(D)$ we define:

Definition. Integral over a Submanifold

For a k -submanifold $M \subset U \subset \mathbb{R}^n$ parametrised by a smooth map from a k -cell $\varphi : D \hookrightarrow U$ with $M = \varphi(D)$:

$$\int_M \omega = \int_{\varphi(D)} \omega := \int_D \varphi^*\omega = \int_D \omega(\varphi(t)) \left(\frac{\partial \varphi}{\partial t_1}, \dots, \frac{\partial \varphi}{\partial t_k} \right) dt_1 \dots dt_k$$

Notice that the $\frac{\partial \varphi}{\partial t_i}$ are *vectors* not just numbers, since φ has multiple output components in general.

3.3 The Exterior Derivative

This is all well and good, and I would recommend making sure you digest that formula well as it is a useful one to be able to just bash with and can be quite satisfying to use. However, the true power of integration is only realised once we can realise its connection with differentiation. That’s where we get powerful results like the fundamental theorem of calculus and its myriad generalisations. For this to work though, we need a notion of a derivative of a k -form. To do this, Let’s think about the simplest possible case first: the derivative of a 0-form, also known as a function to the reals $\mathbb{R}$! just to keep things as simple as humanly possible. Here’s

one thing I want from our notion of derivative: it better agree with the one we already have for functions! That is, this derivative, which I'm suggestively going to call d , should agree with the total derivative D . What kind of thing is Df when $f : \mathbb{R}^n \rightarrow \mathbb{R}$? Well, it takes in an $x \in \mathbb{R}^n$, and spits out a way of linearly mapping tangent vectors to real numbers. That sounds familiar doesn't it! Indeed if $f \in C^\infty(U, \mathbb{R}) = \Omega^0(U)$, then Df is a smooth 1-form, $Df \in \Omega^1(U)$. For 0-forms this will be our definition of df . In particular, notice that if I indicate by $x_i : U \rightarrow \mathbb{R}$ the coordinate projection map onto the x_i -coordinate, then on tangent vectors $dx_i = Dx_i$ is just the coordinate map onto that coordinate in the tangent space.⁵ Notice then that we have the following expansion for any $f \in C^\infty(U, \mathbb{R})$:

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

This suggests another perspective on what d is doing that is more generalisable. The idea is that we measure the change in f along each tangent direction x_i and associate the resulting one dimensional density to that direction, that is, the resulting one-form has that change in f as its dx_i component. We can generalise this! If I have a k -form with its only component f_I being in one basis cotangent vector $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ and all other coefficient functions $f_{I'} = 0$ for $I' \neq I$, then this k -form measures the volume of a tuple of k tangent vectors relative to the coordinates $x_{i_1}, \dots, x_{i_k}$ and multiplied by the density at any point $f_I(x)$. Thinking about tangent vectors as described above as velocities at the point as described in a footnote above, the idea is to measure the amount of the flow described by that collection of velocities multiplied by that density through the parallelepiped described by the tangent directions $x_{i_1}, \dots, x_{i_k}$. The way we take a "derivative" of this to get a 1-higher dimensional form is to imagine extending this parallelepiped in each of the possible other directions, considering the change induced in f_I by nudging along that direction, and call that change the associated density for the one-higher dimensional parallelepiped you get by appending that other direction. Formally, this looks like wedging on one additional dx_j for each of the coordinates x_j , and multiplying by $\frac{\partial f_I}{\partial x_j}$; notice that we will also wedge on the original dx_{i_l} 's already contained in our basis element, but this will make the wedge product zero out and be ignored and thus this respects our idea. This then gives the formula for $f : U \rightarrow \mathbb{R}$, $U \subset \mathbb{R}^n$:

$$d(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

Then extending by linearity, for a general smooth k -form $\omega \in \Omega^k(U)$ with $\omega = \sum_{|I|=k} f_I dx_I$, $U \subset \mathbb{R}^n$ we have the following formula:

$$d\omega = \sum_{|I|=k} \sum_{j=1}^n \frac{\partial f_I}{\partial x_j} dx_j \wedge dx_I = \sum_{i_1 < \cdots < i_k} \sum_{j=1}^n \frac{\partial f_{i_1 \cdots i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

This then gives our definition:

Definition. The Exterior Derivative

The *exterior derivative* is a linear map $d : \Omega^k(U) \rightarrow \Omega^{k+1}(U)$ given for a form $\omega = \sum_{|I|=k} f_I dx_I$ by the formula

$$d\omega = \sum_{|I|=k} \sum_{j=1}^n \frac{\partial f_I}{\partial x_j} dx_j \wedge dx_I = \sum_{i_1 < \cdots < i_k} \sum_{j=1}^n \frac{\partial f_{i_1 \cdots i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

⁵The idea here is that since the derivative is the best linear approximation and the projection map *is linear* then taking the derivative does nothing

Proposition.

- d commutes with pullback, $\varphi^* d\omega = d\varphi^* \omega$
- The *graded product rule*: If $\alpha \in \Omega^k(U), \beta \in \Omega^\ell(U)$, then:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

- d coincides with the total derivative on 0-forms $f \in C^\infty(U, \mathbb{R}), df = Df$.

Note.

- Notice that it follows from our above expansion of df that:

$$d(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

Which can be a nice way to remember things

- Note that if we make the convention that a wedge involving a 0-form is just multiplication – which is not actually the case because this is commutative not anticommutative, but the graded product rule turns out to still hold for this case – then in the case $k = \ell = 0$ the graded product rule reduces to the standard product rule $d(uv) = u dv + v du$.

I think these properties should convince you that d is a reasonably good notion of derivative on forms; in fact it turns out that if you want the above properties for a linear map $\Omega^k(U) \rightarrow \Omega^{k+1}(U)$, you're actually forced into this definition; this follows essentially because you can calculate the formula above purely from the properties of the proposition along with linearity. So now that we've established it's worth considering, let's establish a couple core properties of this exterior derivative; one of them we'll prove, the other we'll defer because the only way I know to prove it involves chain homotopy which suffice to say I don't have time to explain. Don't worry if you're still a bit queasy though; in a second I'm going to make some connections from d to some standard vector calculus operations with which many of you will be familiar, which I think may make it clearer why we care. But for now, I present what is tied for the most important fact about d – the other winner is something Auroux will get to Monday. This is:

Proposition.

$d^2 = 0$, that is, for any form ω , $d(d\omega) = 0$

Proof.

I'll prove this on functions multiplied by pure alternating tensors, the result then follows by extending by linearity. So, consider some $\omega = f dx_I$. We know:

$$d\omega = d(f dx_I) = df \wedge dx_I = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j \wedge dx_I$$

Then applying d again:

$$\begin{aligned}
d^2\omega &= d(d\omega) \\
&= d\left(\sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j \wedge dx_I\right) \\
&= \sum_{k=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_I \\
&= \sum_{1 \leq k < j \leq n} \frac{\partial^2 f}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_I + \sum_{1 \leq k=j \leq n} \frac{\partial^2 f}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_I + \sum_{1 \leq j < k \leq n} \frac{\partial^2 f}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_I \\
&= \sum_{1 \leq k < j \leq n} \frac{\partial^2 f}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_I + \sum_{k=1}^n \frac{\partial^2 f}{\partial x_k \partial x_k} dx_k \wedge dx_k \wedge dx_I + \sum_{1 \leq j < k \leq n} \frac{\partial^2 f}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_I \\
&= \sum_{1 \leq k < j \leq n} \frac{\partial^2 f}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_I + 0 + \sum_{1 \leq j < k \leq n} \frac{\partial^2 f}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_I
\end{aligned}$$

We now use the above proposition that the order of taking partials doesn't matter to swap k and j in the latter sum:

$$\begin{aligned}
d^2\omega &= \sum_{1 \leq k < j \leq n} \frac{\partial^2 f}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_I + \sum_{1 \leq k < j \leq n} \frac{\partial^2 f}{\partial x_k \partial x_j} dx_j \wedge dx_k \wedge dx_I \\
&= \sum_{1 \leq k < j \leq n} \left(\frac{\partial^2 f}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_I + \frac{\partial^2 f}{\partial x_k \partial x_j} dx_j \wedge dx_k \wedge dx_I \right) \\
&= \sum_{1 \leq k < j \leq n} \left(\frac{\partial^2 f}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_I - \frac{\partial^2 f}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_I \right) \\
&= 0
\end{aligned}$$

Thus extending by linearity to a general for $\omega \in \Omega^p(U)$ – I use p rather than k just because in the above proof I used k as an index variable – we indeed retrieve that $d^2 = 0$.

This motivates the following definitions:

Definition.

- A form $\omega \in \Omega^k(U)$ such that $d\omega = 0$ is called *closed*
- A form $\omega \in \Omega^k(U)$ such that there exists $\alpha \in \Omega^{k-1}(U)$ with $\omega = d\alpha$ is called *exact*.

The content of the above proposition is then in this language that all exact forms are closed. In general the converse is not true – you'll come across a wonderful example in the form of the logarithm in complex analysis. There is however a very special and important case where it is, called the *Poincaré Lemma*:

Lemma. Poincaré

If $U \subset \mathbb{R}^n$ is a convex open, then for $1 \leq k \leq n$ we have that $\omega \in \Omega^k(U)$ is closed if and only if it is exact

3.3.1 de Rham Cohomology

I'm putting this in a subsubsection because it's really cool but you also really don't need to know it in this class, to be entirely honest I'm mildly baffled that Auroux includes it at this level but it is cool to have seen. What we have with this sequence of maps $\cdots \xrightarrow{d} \Omega^{k-1}(U) \xrightarrow{d} \Omega^k(U) \xrightarrow{d} \Omega^{k+1}(U) \xrightarrow{d} \cdots$ is a sequence of maps where composing two consecutive ones always yields 0. This should remind you of exact sequences – it's almost the same. In fact, there is exactness at one of these notes exactly when all closed forms are exact, because then $\ker(d : \Omega^k(U) \rightarrow \Omega^{k+1}(U)) = \text{im}(d : \Omega^{k-1}(U) \rightarrow \Omega^k(U))$, rather than just containing it. It turns out a lot of interesting information about the space can be retrieved from studying the degree to which exactness fails. This is measured by the size of the k 'th *de Rham Cohomology Group*:⁶

Definition. de Rham Cohomology

The k 'th *de Rham Cohomology group* of a smooth manifold M is defined to be, indicating by d_k the exterior derivative on k forms:

$$H_{dR}^k(M) := \ker d_k / \text{im } d_{k-1}$$

So the statement of the Poincaré Lemma is that convex open subsets of $\mathbb{R}^n$ have trivial de-Rham cohomology, $H_{dR}^k(U) = 0$ for all $k > 0$. The de-Rham cohomology group is a very important tool in the topology of smooth manifolds, and has a lot to do with how many “holes” the space has; I'll note that $H_{dR}^1(S^1) = \mathbb{R}$, $H_{dR}^1(S^1 \times S^1) = \mathbb{R}^2$, $H_{dR}^1(\mathbb{R}^n) = 0$, $H_{dR}^n(S^n) = \mathbb{R}$ for $n > 0$, and let you ponder those.

3.4 Connections to standard Vector Calculus

Having delved way into abstraction, I'd now like to connect the exterior derivative d to some possibly more familiar concepts from vector calculus. In particular, I'm going to be dealing with the standard operations on vector fields in $\mathbb{R}^3$, with coordinates x, y, z and corresponding basis vectors $\hat{i}, \hat{j}, \hat{k}$. Recall the following operations:

Definition. Gradient, Curl, Divergence

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a smooth scalar field on $\mathbb{R}^3$, and $\vec{F} = (P, Q, R) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a smooth vector field on $\mathbb{R}^3$. We then define:

- The *gradient* of f :

$$\text{grad } f := \nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$$

- The *curl* of $\vec{F}$:

$$\text{curl } \vec{F} := \nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}$$

⁶Really it's a vector space since $\Omega^k(U)$ is a vector space

- The *divergence* of $\vec{F}$ is:

$$\operatorname{div} \vec{F} := \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

I claim that each of these is just a special instance of the exterior derivative, up to a change of basis to pass between the basis $\hat{i}, \hat{j}, \hat{k}$ of the tangent space and bases for various exterior powers of the cotangent space. In particular, what you'll come to see is that the reason $\mathbb{R}^3$ admits all these nice differential operators – in particular curl – is that the dimension of $\wedge^2 \mathbb{R}^3$ coincides with that of $\wedge^1 \mathbb{R}^3 \cong \mathbb{R}^3$. Throughout I'm going to be making use of a certain duality principle called *hodge duality*. You don't need to understand it fully, all you need to know is that there exist in the case of $M = \mathbb{R}^3$ isomorphisms $\mathbb{R} \cong \wedge^3 T_p^*$ and $T_p^* \cong \wedge^2 T_p^*$ both indicated by $\star$ with the property that $\star 1 = dx \wedge dy \wedge dz$, $\star dx = dy \wedge dz$, $\star dy = dz \wedge dx$, $\star dz = dx \wedge dy$, and $\star^2 = \operatorname{id}$. We're gonna pass through this isomorphism a couple times, but just keep in mind that these are just exchanging the standard bases for these spaces, and so won't do anything to the coordinate representations. I'm also going to be using the fact that the most obvious way in coordinates to associate a vector field $P\hat{i} + Q\hat{j} + R\hat{k}$ to a 1-form is to map it to $Pdx + Qdy + Rdz$. Indicate this isomorphism with $\flat : T_p \rightarrow T_p^*$ and its inverse by $\sharp : T_p^* \rightarrow T_p$ – this is amusingly enough actually very close to standard notation, normally we'd be putting the $\flat$ in a superscript but I don't like that notation for this. Then observe we have (ignore the $\star$, $\flat$, and $\sharp$ symbols here, they're just formal isomorphisms, the core content is in ignoring them):

$$\begin{aligned} df &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \\ &= \flat \operatorname{grad} f \\ \flat \vec{F} &= Pdx + Qdy + Rdz \\ d\flat \vec{F} &= \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial P}{\partial z} dz \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy + \frac{\partial Q}{\partial z} dz \wedge dy + \frac{\partial R}{\partial x} dx \wedge dz + \frac{\partial R}{\partial y} dy \wedge dz \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy \\ &= \star \left(\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dx + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dy + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dz \right) \\ &= \star \flat \operatorname{curl} \vec{F} \\ \star \flat \vec{F} &= Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy \\ d\star \flat \vec{F} &= \frac{\partial P}{\partial x} dx \wedge dy \wedge dz + \frac{\partial Q}{\partial y} dy \wedge dz \wedge dx + \frac{\partial R}{\partial z} dz \wedge dx \wedge dy \\ &= \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz \\ &= \star \operatorname{div} \vec{F} \end{aligned}$$

So in summary we have:

Proposition.

1. $\operatorname{grad} f = \sharp df$
2. $\operatorname{curl} \vec{F} = \sharp \star d(\flat \vec{F})$
3. $\operatorname{div} \vec{F} = \star d(\star \flat \vec{F})$

So up to a couple purely formal factors to make the kinds of things we're dealing with line up, grad, curl, and div are all just instances of the exterior derivative! In particular:

$$\begin{aligned}\text{curlgrad}f &= \text{curl}\sharp df = \sharp \star d\flat\sharp df = \sharp \star d^2f = 0 \\ \text{divcurl}\vec{F} &= \text{div}\sharp \star d(\flat\vec{F}) = \star d \star \flat\sharp \star d(\flat\vec{F}) = \star d \star^2 d(\flat\vec{F}) = \star d^2(\flat\vec{F}) = 0\end{aligned}$$

So we have proven two neat little multivariable calculus propositions:

Proposition.

- $\text{curl} \circ \text{grad} = 0$
- $\text{div} \circ \text{curl} = 0$

This gives you a taste of how many of the classic theorems of vector calculus are, once you do the right interpreting work, just instantiations of more general theorems in the theory of differential forms. The grandest example of this principle is something we'll cover next lecture, but I just can't help myself from sharing it now. If we define the boundary ∂M of a domain of integration M to be the image of the obvious boundary of the k -cell by which it is parametrised, then we have:

Theorem. Stokes' Theorem

For $M \subset \mathbb{R}^n$, $\omega \in \Omega^k(M)$, we have:

$$\int_{\partial M} \omega = \int_M d\omega$$

Corollary.

- Green's Theorem
- The vector calculus version of Stokes' Theorem
- The Divergence Theorem
- The fundamental theorem of line integrals
- The fundamental theorem of calculus

Congratulations! If you're willing to believe the proof sketch Auroux gives of Stokes' theorem next time, you will have done all the theorems of multivariable calculus.